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Mathematical Reviews

Vol. 20, No. 10 NOVEMBER, 1959 Reviews 6345-6973d

LOGIC AND FOUNDATIONS

See also 6963.

6345:

Falevič, B. Ya. A new method of proving incompleteness theorems for systems with Carnap rule, and its application to the problem of interrelation between classical and constructive analyses. Dokl. Akad. Nauk SSSR 120 (1958), 1210–1213. (Russian)

6346:

Takeuti, Gaisi. Ordinal diagrams. J. Math. Soc. Japan. 9 (1957), 386–394.

Ordinal diagrams of order n, which are defined here, are a notion closely related to the proof-figures of G. Gentzen. With this in mind, the author defines a relevant linear ordering of these and examines its properties, particularly with regard to transfinite induction. The latter is of interest because of its use in consistency proofs. *L. N. Gál* (New Haven, Conn.)

6347:

Takeuti, Gaisi. Remark on the fundamental conjecture of GLC. J. Math. Soc. Japan 10 (1958), 44–45.

GLC is a generalization of Gentzen's LK to include bound and free variables of higher order. The fundamental conjecture referred to in the title states that every provable sequent of GLC is provable without cut. The present note contains two propositions which clarify some points raised by the author [same J. 7 (1955), 249–275; MR 18, 1]. In particular, the author explains what he means by "analysis" and shows that if the fundamental conjecture of GLC holds, then analysis is consistent.

L. N. Gál (New Haven, Conn.)

6348:

Takeuti, Gaisi. On the fundamental conjecture of GLC. V. J. Math. Soc. Japan 10 (1958), 121–134.

This is the fifth in a sequence of notes reporting on attempts to prove the fundamental conjecture of GLC [parts I to IV were in same J. 7 (1955), 249–275, 394–408; 8 (1956), 54–64, 145–155; MR 18, 1; and see above review]; it is a generalization of part IV. It introduces the notion of regular proof-figures of GLC and shows that every sequent provable by a regular proof-figure is provable without cut. The proof makes use of the results of the author's paper on 'ordinal diagrams' [#6346 above].

L. N. Gál (New Haven, Conn.)

6349:

Lorenzen, Paul. Über die Syllogismen als Relationenmultiplikationen. Arch. Math. Logik Grundlagenforsch. 3 (1957), 112–116.

Classical syllogistic is regarded as a calculus of binary relations, a syllogism corresponding to an equation of form $\rho\sigma=\tau$ in this calculus. Here ρ, σ, and τ have as range, Σ, the set of relations a (universal affirmative judgment), i (particular affirmative judgement), e (universal negative judgment), and o (particular negative judgment) of traditional logic as well as the converses of these relations. Thus "Calemes" corresponds to $ae=e$, "Darapti" to $i=\breve{a}a$. (The converse of ρ is $\breve{\rho}$; $\breve{i}$ is i, $\breve{e}$ is e.) The product of relations in Σ as defined is essentially the familiar relative product and hence associative, but Σ is not closed under this multiplication. Closure is obtained by adjoining the universal relation to Σ, so that Σ may be regarded as embedded in a semi-group Σ_7 (without cancellation) consisting of seven elements. An alternative nine-member semi-group extension of Σ, Σ_9, is specified; Σ_9 has a simple algebraic connection with Σ_7. Throughout the paper the question of deriving the various equations from certain axioms is discussed. [Cf. B. Baron von Freytag Löringhoff, Z. für Philosoph. Forsch. 4 (1950), 235–256; *Logik*, Kohlhammer, Stuttgart, 1955; and H. Gericke, Arch. Math. 3 (1952), 421–433; MR 14, 935.].

This article should certainly be pointed out to those beginning students of modern logic or algebra who are familiar with traditional logic.

W. W. Boone (Urbana, Ill.)

6350:

Scott, D.; and Tarski, A. The sentential calculus with infinitely long expressions. Colloq. Math. 6 (1958), 165–170.

For any ordinal numbers α and β, the class of α-well-formed formulas (α-wffs) is the least class of formulas closed under the following rules: (i) a variable p_ξ, where $\xi<\beta$, is an α-wff; (ii) if A and B are α-wffs, then so are $[A\to B]$ and $\sim A$; (iii) if $A_0, A_1, \cdots, A_\xi, \cdots$ is a well-ordered sequence of type less than α of α-wffs, then

$$\bigwedge[A_0A_1\cdots A_\xi\cdots] \text{ and } \bigvee[A_0A_1\cdots A_\xi\cdots]$$

are α-wffs.

The result of substituting α-wffs for the variables of a theorem of the ordinary sentential calculus is an α-theorem; and if $A_0, A_1, \cdots, A_\xi, \cdots$ is a well-ordered sequence of type γ, $\gamma<\alpha$, of α-wffs and B is an α-wff, then

$$[\bigwedge[A_0A_1\cdots A_\xi\cdots]\to A_\eta] \text{ and } [A_\eta\to\bigvee[A_0A_1\cdots A_\xi\cdots]]$$

are α-theorems for each $\eta<\gamma$; and if $[B\to A_\eta]$ (respectively $[A_\eta\to B]$) is an α-theorem for all $\eta<\gamma$, then $[B\to\bigwedge[A_0A_1\cdots A_\xi\cdots]]$ (respectively $[\bigvee[A_0A_1\cdots A_\xi\cdots]\to B]$) is an α-theorem.

An outline of the completeness theorem for $\alpha\leq\omega_1$ is given and it is shown that the concept of α-theorem is not complete for all $\alpha>\omega_1$. A modification of the definition of "α-theorem" permits the completeness theorem to be proved for all α.

P. C. Gilmore (Yorktown Heights, N.Y.)

6351:

Tarski, A. Remarks on predicate logic with infinitely long expressions. Colloq. Math. 6 (1958), 171–176.

An atomic wff of the predicate logic P_1 with denumerably long expressions is a sequence consisting of a predicate letter and a finite sequence of variables, one of the predicate letters being binary indentity. A wff of P_1 is an $(\omega+1)$-wff (see above review) in which the atomic wff are

those of P_1, or any sequence

$$(\forall v_0 \cdots v_\xi \cdots) F \text{ or } (\exists v_0 \cdots v_\xi \cdots) F,$$

where F is any wff of P_1 and $v_0, \cdots, v_\xi, \cdots$ is a finite or denumerable sequence of variables.

Provability in P_1 is not defined, but several model-theoretic results are stated without proof.

P. C. Gilmore (Yorktown Heights, N.Y.)

6352:

Sikorski, R. On Herbrand's theorem. Colloq. Math. **6** (1958), 55–58.

The Herbrand Theorem asserts that with every formula α_0 (of the classical predicate calculus) in the normal prenex form we can associate an infinite sequence $\alpha_1, \alpha_2, \cdots$ of formulas without quantifiers such that α_0 is provable if and only if one of the formulas α_n $(n=1, 2, \cdots)$ is provable. The α_n have the form of partial disjunctions of an infinite disjunction, and the provability of some α_n suggests reduction of an infinite "covering" to a finite "covering", which in turn suggests the Heine-Borel Theorem. The author gives a topological proof of one-half of the Herbrand Theorem, based on a representation theorem of Rieger [Fund. Math. **38** (1951), 35–52; MR **14**, 347] and making direct use of the Heine-Borel Theorem.

B. A. Galler (Ann Arbor, Mich.)

6353:

Rasiowa, H.; and Sikorski, R. On the isomorphism of Lindenbaum algebras with fields of sets. Colloq. Math. **5** (1958), 143–158.

If S is a logical system based on the first order predicate calculus, then a choice of a set A of certain well-formed formulas to be axioms yields an elementary theory $S(A)$. If $C_n(A)$ denotes the theorems of the theory, one can define an equivalence relation of the set W of well-formed formulas of S: $a \sim b$ if and only if $a \rightarrow b$ and $b \rightarrow a$ are both in $C_n(A)$ for all $a, b \in W$. The Lindenbaum algebra $L(A)$ of $S(A)$ is the algebra obtained by regarding W as an abstract algebra and forming the equivalence classes relative to the equivalence just described. This paper describes conditions under which $L(A)$ is representable as a field of sets. A theorem due to Rieger [Fund. Math. **38** (1951), 35–52; MR **14**, 347] is generalized to the case card $W > \aleph_0$.

B. A. Galler (Ann Arbor, Mich.)

6354:

Takeuti, Gaisi. On the theory of ordinal numbers. II. J. Math. Soc. Japan **10** (1958), 106–120.

In Part I [same J. **9** (1957), 93–113; MR **19**, 237], Takeuti presents an axiomatic theory of ordinals and shows that Gödel's system of set theory is consistent if this theory of ordinals is. The present paper shows that the converse of this is also true.

L. N. Gál (New Haven, Conn.)

6355:

Friedberg, Richard. Un contre-exemple relatif aux fonctionnelles récursives. C. R. Acad. Sci. Paris **247** (1958), 852–854.

Let F be the set of Gödel numbers (G.N.) of recursive functions. By a simple construction the author obtains a recursively enumerable set R of natural numbers with the following properties: (1) If R contains a G.N. of a recursive function φ, then R contains all G.N.'s of φ, (2) there is no partial recursive function Ψ with domain $\mathscr{D}\Psi$ such that, for the set G of G.N.'s of elements of $\mathscr{D}\Psi$, $G \cup F = R \cup F$. The construction effected in the proof also yields an example of an effective operation defined for all recursive functions and with partial recursive functions as values, which is not the restriction of a partial recursive functional. This solves negatively a problem of Myhill and Shepherdson [Z. Math. Logik Grundlagen Math. **1** (1955), 310–317; MR **17**, 1039].

A. Heyting (Amsterdam)

6356:

Nagornyĭ, N. M. A minimal alphabet of algorithms over a given alphabet. Trudy Mat. Inst. Steklov. **52** (1958), 66–74. (Russian)

Every normal algorithm (in the sense of Markov) over an alphabet A is equivalent with respect to A to an algorithm in the alphabet $A \cup \{\alpha\}$, where $\alpha \notin A$.

A. Heyting (Amsterdam)

6357:

Orlovskiĭ, È. S. Some questions in the theory of algorithms. Trudy Mat. Inst. Steklov. **52** (1958), 140–171. (Russian)

In the first part of the paper the converse of an algorithm is introduced. Let $\mathfrak{A}$ be an algorithm in the alphabet B and $\mathfrak{M}$ an algorithm over B. The algorithm $\mathfrak{C}$ is the converse of $\mathfrak{A}$ for the words which are nullified by $\mathfrak{M}$, if (1) $\mathfrak{C}$ is applicable to a word P in B if and only if there is a word Q in B such that $\mathfrak{A}(Q)=P$ and $\mathfrak{M}(Q)=\Lambda$ (the empty word), (2) if $\mathfrak{C}(P)$ exists, then $\mathfrak{M}(\mathfrak{C}(P))=\Lambda$ and $\mathfrak{A}(\mathfrak{C}(P))=P$. The construction of $\mathfrak{C}$ from $\mathfrak{A}$ and $\mathfrak{M}$ is given in detail. In the second part, an explicit scheme, consisting of $10n^2+29n+44$ formulas, where n is the number of letters in B, is given for a universal algorithm in the sense of Markov [Teoriya algorifmov, Trudy Mat. Inst. Steklov no. 42, Izdat. Akad. Nauk SSSR, Moscow, 1954; MR **17**, 1038; Ch. IV].

A. Heyting (Amsterdam)

6358:

Ceĭtin, G. S. An associative calculus with an insoluble problem of equivalence. Trudy Mat. Inst. Steklov. **52** (1958), 172–189. (Russian)

The word problem for groups can be immediately reduced to that for semigroups with defining relations of the form $A \rightarrow$. Thus, by Novikov's theorem, the latter is recursively unsolvable. (The author remarks that it would be interesting to have a direct proof of this result.) In its turn, this problem is reduced to the word problem in the semigroup with generators a, b, c, d, e under defining relations $ac \leftrightarrow ca$, $ad \leftrightarrow da$, $bc \leftrightarrow cb$, $bd \leftrightarrow db$, $eca \leftrightarrow ce$, $edb \leftrightarrow de$, $cca \leftrightarrow ccae$; consequently, this problem is recursively unsolvable. By an analogous method it is proved that for the semi-group with generators a, b, c, d, e and as defining relations the first six above with $cdca \leftrightarrow cdcae$, $caaa \leftrightarrow aaa$, $daaa \leftrightarrow aaa$ the problem of equivalence with aaa is recursively unsolvable.

A. Heyting (Amsterdam)

6359:

Rogers, Hartley, Jr. The present theory of Turing machine computability. J. Soc. Indust. Appl. Math. **7** (1959), 114–130.

This is an expository paper, the text of an invited address to SIAM, giving a short account of some of the key results in recursive function theory and a statement of several questions current in the subject.

R. M. Baer (Berkeley, Calif.)

6360:

Heyting, A. Blick von der intuitionistischen Warte. Dialectica **12** (1958), 332–345. (French and English summaries)

"The paper contains remarks on intuitionism and its relations with other domains of foundational research.

Inside the intuitionistic mathematics, in connection with Griss' criticism against the use of negation, different degrees of evidence are distinguished, depending upon the way in which conditioned constructions are admitted. Some difficulties in the theory of finite species are discussed. Concerning the foundational research in general it is observed that it has separated intuitive, formal and platonistic constituents in classical mathematics. Some remarks are made on Church's thesis in the theory of recursive functions." (Author's abstract)

P. C. Gilmore (Yorktown Heights, N.Y.)

6361:

Vorob'ev, N. N. A new algorithm of derivability in a constructive calculus of statements. Trudy Mat. Inst. Steklov. **52** (1958), 193–225. (Russian)

Let a formula in the propositional calculus be called simple if it is either a variable or a negation or an implication, let it be called standard if it is a disjunction of conjunctions of simple formulas, and let it be called normal if in every subformula of the form $P\to Q$, P is simple and Q is standard. The author shows that in the intuitionistic propositional calculus I every formula is equivalent to a normal formula. He constructs a Gentzen form calculus J, which is equivalent to I, proves the theorem corresponding to Gentzen's Hauptsatz, and shows how the decision procedure for J can be much simplified by the restriction to normal formulas.

A. Heyting (Amsterdam)

6362:

Bernays, Paul. Betrachtungen zum Paradoxon von Thoralf Skolem. Avh. Norske Vid. Akad. Oslo. I. **1957**, no. 5, 9 pp.

The opinion of the author on the matter could be summarised as follows.

The so-called Skolem's antinomy (concerning countable models of the axiomatic set theory) is not a decisive argument against the usual conception of analysis within the (axiomatic) set theory. The disadvantage of the purely operational viewpoint is not only that it brings unnecessary complications but also that it neglects the fruitful geometrical idea of the homogeneity of the continuum.

The second number class presents a proper uncountable frame of mathematical thinking no matter whether constructive or not (if not arbitrarily limited).

L. Rieger (Prague)

6363:

Maehara, Shôji. Über die rekursive Einführung der Funktionen in der reinen Zahlentheorie. Proc. Japan Acad. 33 (1957), 111–113.

Der Gentzensche Beweis der Widerspruchsfreiheit (Wf.) der Arithmetik setzt voraus, dass die benutzten primitiven Funktionen entscheidbar sind. Durch Bildung von ι-Termen entstehen in diesen Arithmetik auch nicht-entscheidbare Funktionen. Verf. stellt die wichtige Frage, ob daher die Benutzung des Rekursionsschemas

$$\phi(1)=b,\ \phi(n+1)=B(n, \phi(n))$$

noch wf. bleibt, wenn rechts evtl. ι-Terme stehen. Er beantwortet diese Frage positiv, indem er auf die nach der Gentzenschen Methode beweisbare Wf. der verzweigten Typentheorie zurückgreift. Wie schon bei Dedekind steht, kann man nämlich für die Formel

$$R(\varphi, a)\equiv[\varphi(1)=b\wedge\bigwedge_n(n<a\to\varphi(n+1) =B(n, \varphi(n))\wedge\bigwedge_n(n>a\to\varphi(n)=1)]$$

beweisen: $\bigwedge_a\bigvee_\varphi R(\varphi, a)$ und $R(\varphi, a)\wedge R(\psi, a)\to\varphi=\psi$.

Diese Beweise erfordern nur die erste Schicht der verzweigten Typentheorie. Setzt man dann $\phi(n)=y\equiv \bigwedge_\varphi(R(\varphi, n)\to\varphi(n)=y)$, so erhält man die Rekursionsgleichungen.

P. Lorenzen (Kiel)

6364:

Maehara, Shôji. Remark on Skolem's theorem concerning the impossibility of characterization of the natural number sequence. Proc. Japan Acad. 33 (1957), 588–590.

Vom Gödelschen Vollständigkeitssatz ausgehend folgt der Skolemsche Satz von der Polymorphie jeder axiomatisierten Arithmetik zwar unmittelbar aus dem Gödelschen Unvollständigkeitssatz — aber nur unter der einschränkenden Bedingung, dass ein rekursiv-aufzählbares Axiomensystem vorliegt. Den uneingeschränkten Skolemschen Polymorphiesatz erhält man dagegen, wenn man zu einem beliebigen Axiomensystem mit einer neuen Konstanten u hinzufügt: $u\neq0$, $u\neq1$, $u\neq2$, $\cdots$. Weil noch jedes endliche Teilsystem konsistent ist, liefert der Vollständigkeitssatz die Existenz eines (Nicht-Standard-) Modelles.

Dieser Beweis findet sich schon bei A. Robinson [On the metamathematics of algebra, North Holland Publ. Co, Amsterdam, 1951; MR **13**, 715].

P. Lorenzen (Kiel)

SET THEORY

See 6478.

COMBINATORIAL ANALYSIS

6365:

Clarke, L. E.; and Singer, James. On circular permutations. Amer. Math. Monthly **65** (1958), 609–610.

The authors establish in a manner independent of Pólya's theorem the formula

$$P=\sum_{d\mid n}(1/n)\phi(d)F_d,$$

where $F_d=(n/d)\prod(r_i/d)^{-1}$, for the number of circular permutations of n objects, r_1 of type $a_1, \cdots, r_p$ of type a_p. {This result appeared almost simultaneously as problem 37(b), p. 162, John Riordan, ***Introduction to combinatorial analysis***, Wiley, New York, 1958 [MR **20** #3077], with a reference to E. Lucas, ***Théorie des nombres***, Paris, 1891; pp. 501–503, "from C. Moreau".}

R. L. Davis (Charlottesville, Va.)

ORDER

6366:

Jakubík, Ján. Note on the endomorphisms of lattices. Časopis Pěst. Mat. 83 (1958), 226–229. (Slovak. Russian and English summaries)

Eine Abbildung f eines Verbandes S in sich wird bekanntlich als ein $\cup$-Endomorphismus auf S bezeichnet, wenn

$$x, y\in S\Rightarrow f(x)\cup f(y)=f(x\cup y).$$

Sei E die von allen $\cup$-Endomorphismen auf S gebildete und durch die folgende Festsetzung teilweise geordnete Menge: Für $f, g \in E$ gilt $f \leq g$ genau dann, wenn $x \in S \Rightarrow f(x) \leq g(x)$. In der vorgelegten Arbeit werden folgende Sätze bewiesen: 1. Ist der Verband S vollständig, so stellt auch E einen vollständigen Verband dar. 2. Die teilweise geordnete Menge E braucht kein semimodularer Verband zu sein, selbst dann nicht, wenn der Verband S endlich ist. Diese Sätze lösen das Problem 93 von Birkhoff [Lattice theory, Amer. Math. Soc., New York, 1948; MR **10**, 673; S. 209]. Ausserdem stellt der Verf. fest, dass die Behauptung [l.c. S. 208] nach der die $\cup$-Endomorphismen auf einem Verband L eine l-Halbgruppe darstellen, nicht zutrifft. *O. Borůvka* (Brno)

6367:

Hartmanis, Juris. Lattice theory of generalized partitions. Canad. J. Math. **11** (1959), 97–106.

A generalized partition of type $n \geqq 1$ is a collection of subsets of a set S such that any n distinct elements of S are contained in exactly one subset and every subset contains at least n distinct elements. For $n=1$, this gives ordinary equivalence relations; for $n=2$, the geometries previously studied by the author, when the lines defining the geometry are the subsets in the partitions. The general partitions form a complete complemented point lattice; and it is isomorphic to the lattice of subspaces of the same geometry. The author characterizes the lattice when $n=2$. The general partition lattice has only automorphisms corresponding to permutations of the basic set S. *O. Ore* (New Haven, Conn.)

6368:

Kolibiar, Milan. Une opération ternaire dans les treillis. Czechoslovak Math. J. **6(81)** (1956), 318–329. (Russian. French summary)

Es sei S ein Verband mit den Operationen $\cap$, $\cup$. Für jedes Tripel $[a, b, c]$ der Elemente von S, für welches

$$(a\cap b)\cup(b\cap c)\cup(c\cap a)=(a\cup b)\cap(b\cup c)\cap(c\cup a)$$

ist, wird

$$(a, b, c)=(a\cap b)\cup(b\cap c)\cup(c\cap a)$$

gesetzt; damit ist eine ternäre Operation definiert. Es sei t ein neutrales Element von S, welches in jedem Intervall $[a, b]$, in dem es liegt, ein Komplement a^*b hat; t wird Element mit der Eigenschaft (c) heißen. Für beliebige Elemente a, b in S setzen wir

$$a\wedge b=(a, t, b),\ a\vee b=(t\cap a\cap b)^*(t\cup a\cup b).$$

Dann ist die Menge S mit den Operationen $\wedge$, $\vee$ wieder ein Verband, der mit S_t bezeichnet wird.

Für ein Paar S, S' von Verbänden, die auf derselben Menge definiert sind, werden u.a. folgende Eigenschaften untersucht: (E) Es gibt ein Element t mit der Eigenschaft (c) in S, für welches $S'=S_t$ ist; (B) konvexe Unterverbände von S und S' sind dieselben; (O) die ternäre Operation (a, b, c) wird in den Verbänden S, S' für dieselben Tripel von Elementen definiert und erreicht in S und S' dieselben Werte.

Es wird gezeigt, daß für Verbände mit einem größten und einem kleinsten Element die Eigenschaften (E), (B), (O) äquivalent sind; für den allgemeinen Fall wird ein schwächeres Resultat abgeleitet. *M. Novotný* (Brno)

6369:

Jakubík, Ján. Sur les axiomes des multistructures. Czechoslovak Math. J. **6(81)** (1956), 426–430. (Russian. French summary)

M. Benado [derselbe J. **5(80)** (1955), 308–344; MR **17**, 937] hat einen Multiverband (multistructure) als eine Menge M definiert, in der zu jedem geordneten Paar von Elementen a, b zwei Teilmengen $a\wedge b$ und $a\vee b$ von M zugeordnet sind, welche einem gewissen Axiomensystem genügen. An einem Beispiel wird gezeigt, daß das letzte Axiom von den übrigen nicht abhängt. Es wird ferner bewiesen, daß es Multiverbände gibt, in denen die Operationen assoziativ sind, die aber keine Verbände sind. Sind aber die Mengen $a\wedge b$, $a\vee b$ immer nicht leer und die Operationen assoziativ, so ist der Multiverband ein Verband. Damit sind zwei von Benado formulierte Probleme gelöst. *M. Novotný* (Brno)

6370:

Benado, Mihail. Sur la fonction de Möbius. C. R. Acad. Sci. Paris **246** (1958), 863–865.

The author states some properties of distributive multistructures [Benado, Czechoslovak Math. J. **5(80)** (1955), 308–344; MR **17**, 937], concerned with the existence of elements in certain quotients and elements of certain subsets of a (multistructure-) meet or join. If the Möbius function μ on a hierarchy H (p.o. set with all quotients finite) satisfies $\mu(a)\mu(b)=\mu(d)\mu(m)$ for all $d \in a\cup b$, $m \in a\cap b$, then H is called a Möbius hierarchy. Some characteristics of μ in a Möbius hierarchy are determined. In any distributive hierarchy, μ is multiplicative. Any hierarchy, with 0, which can be decomposed into a cartesian product of an ordered set of elementary hierarchies must be a Möbius hierarchy. {The author's citation of prerequisite material is inadequate; indeed, this appears in part in a subsequent paper [#6371 below].} *P. M. Whitman* (Baltimore, Md.)

6371:

Benado, Mihail. Sur la fonction de Möbius. C. R. Acad. Sci. Paris **246** (1958), 2553–2555.

The author announces results concerned with the vanishing of a summing of the Möbius function [see Birkhoff, Lattice theory, revised ed., Amer. Math. Soc., New York, 1948; MR **10**, 673; pp. 14–15] over certain subsets of X; here X is a partially ordered set which has a first element and which satisfies the condition that all subsets of the form $[x:a \leqq x \leqq b]$ are finite. The author also announces results for overture operators (dual to closure operators) on X for the case in which X satisfies not only the conditions above, but in addition has the property that whenever a subset of X has an upper bound z the subset has a (not necessarily unique) minimal upper bound y such that $y \leqq z$. *R. M. Baer* (Berkeley, Calif.)

6372:

Avann, S. P. Dual symmetry of projective sets in a finite modular lattice. Trans. Amer. Math. Soc. **89** (1958), 541–558.

A theorem of the reviewer [Ann. of Math. (2) **60** (1954), 359–364; MR **16**, 106] states that in a finite modular lattice the number of elements covered by precisely k elements is equal to the number of elements covering precisely k elements. In this paper the author uses the methods introduced by the reviewer to obtain the following generalization. Let $Q_1, \cdots, Q_n$ be classes of projective complemented quotients of a modular lattice. Then the number of elements which are denominators of precisely k_i quotients of Q_i for each i is equal to the number of elements which are the numerators of precisely k_i quotients of Q_i for each i. If the Q_i are the classes of projective prime quotients, this theorem reduces to the one stated above. *R. P. Dilworth* (Pasadena, Calif.)

6373:

Szász, G. Note on complemented modular lattices of finite length. Acta Sci. Math. Szeged **19** (1958), 224–228.

Necessary and sufficient conditions are given that a finite dimensional complemented modular lattice contain three distinct elements whose pairwise joins are the unit element and whose pairwise meets are the null element of the lattice. *R. P. Dilworth* (Pasadena, Calif.)

6374:

Fryer, K. D.; and Halperin, Israel. On the construction of coordinates for non-Desarguesian complemented modular lattices. I, II. Nederl. Akad. Wetensch. Proc. Ser. A **61**=Indag. Math. **20** (1958), 142–161.

The authors continue in these papers their program of simplifying and extending the coordinatization theory of complemented modular lattices initiated by J. von Neumann. If L is a complemented modular lattice having a normalized 3-frame satisfying certain uniqueness and invariance conditions, it is shown that associated with L is an alternative regular ring R. Such a ring is characterized by the properties: 1) $(xy)z=x(yz)$ if at least one of x, y, z, xy, yz is idempotent; 2) xy is idempotent if $x=x(yx)$ and yx is idempotent; 3) for each x in R there exists an idempotent e and an element y in R such that $xe=x$, $ey=y$, $yx=e$. This theorem includes the Moufang coordinatization of projective planes satisfying the "uniqueness of harmonic conjugate point" conditions.
R. P. Dilworth (Pasadena, Calif.)

6375:

Hukuhara, Masuo. ⊕-endomorphisme et ∩-endomorphisme d'un treillis en dualité et la théorie de Riesz sur l'endomorphisme complètement continu. II. Extension de la théorie de Riesz aux applications du treillis. Funkcial. Ekvac. **1** (1958), 103–120. (Esperanto summary)

[For part I see Funkcial. Ekvac. **1** (1958), 85–102; MR **20** #3083.] Let L be a modular lattice with least element 0 and greatest element 1; let λ be a mapping of L into itself such that
$$\lambda(x\vee y)=\lambda(x)\vee\lambda(y);$$
and let λ^* be a mapping such that
$$\lambda^*(x\wedge y)=\lambda^*(x)\wedge\lambda^*(y).$$
These mappings are said to be dual if, for any $x\in L$,
$$\lambda(\lambda^*(x))=x\wedge\lambda(1),\quad \lambda^*(\lambda(x))=x\vee\lambda^*(0).$$
The author exhibits a natural ordering for such mappings, shows uniqueness of the dual (when it exists), and gives results (corresponding to known results for the lattice of subspaces of a linear space) on the properties of iterations of such mappings λ for the cases both when λ and λ^* do, and do not, commute. *R. M. Baer* (Berkeley, Calif.)

6376:

Grätzer, G.; and Schmidt, E. T. Characterizations of relatively complemented distributive lattices. Publ. Math. Debrecen **5** (1958), 275–287.

The authors show that any distributive lattice which is not relatively complemented has a homomorphic image isomorphic to the three-element chain. Systematic use of this result gives some new proofs of known characterizations of relatively complemented distributive lattices.
R. P. Dilworth (Pasadena, Calif.)

6377:

Ellis, David. Remarks on the elementary symmetric functions. Math. Mag. **32** (1958), 75–78.

The author draws attention to various identities involving elementary symmetric functions (say of $X_1, \cdots, X_n$). He then discusses some properties of these functions when $X_1, \cdots, X_n$ are elements of a Hilbert space. Finally, he considers analogous expressions for the case when the X's are elements of a distributive lattice (sums and products being replaced by unions and intersections, respectively), and obtains a number of results concerning chains in lattices. *L. Mirsky* (Sheffield)

6378:

Jakubík, Ján. The center of infinitely distributive lattices. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **7** (1957), 116–120. (Slovak. Russian summary)

Es werden Verbände mit nichtleerem Zentrum studiert, die folgende Eigenschaft (A) haben: Für jede Teilmenge M des Zentrums, für welche sup M und inf M existieren, liegen diese Elemente im Zentrum. Es wird bewiesen, daß unendlich distributive vollständige Verbände die Eigenschaft (A) haben. Es werden Beispiele eines distributiven Verbandes und eines vollständigen Verbandes konstruiert, die die Eigenschaft (A) nicht haben. Ein Satz über Zerlegungen eines unendlich distributiven vollständigen Verbandes in direkte Faktoren wird bewiesen. *M. Novotný* (Brno)

6379:

Grätzer, G.; and Schmidt, E. T. Two notes on lattice-congruences. Ann. Univ. Sci. Budapest. Eötvös. Sect. Math. **1** (1958), 83–87.

This note contains a proof that any complete, weakly-atomic Boolean algebra or chain is the lattice of all convergence relations of a suitable lattice.
R. P. Dilworth (Pasadena, Calif.)

6380:

Dwinger, Ph. On the completeness of the quotient algebras of a complete Boolean algebra. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. **62**=Indag. Math. **21** (1959), 26–35.

Let A be an infinite, complete Boolean algebra, I an ideal of A and $\bar{I}$ the principal ideal generated by I.

In part I it is shown that there exists I such that A/I is not complete. In part II it is shown that A/I is complete if and only if $\bar{I}/I$ is complete. From this it follows that if I is a dense ideal of A such that the sum of every infinite disjointed subset of I does not belong to I, then A/I is not complete. *J. Hartmanis* (Schenectady, N.Y.)

GENERAL ALGEBRAIC SYSTEMS

See 6369.

THEORY OF NUMBERS

See also 6464, 6488, 6507, 6513.

6381:

***Kaprekar, D. R. Puzzles of the self-numbers.** Published by the author, Devlali, 1959. 24 pp. Re. 1.50.

If x is any positive integer, let t_x denote the sum of the digits in the decimal representation of x and put $x_1=x+t_x$. Similarly define $x_2=x_1+t_{x_1}$, and so on. We thus obtain a sequence of integers $x_0=x, x_1, x_2, \cdots$ which the

writer calls the digitadition series of x; this is denoted by the symbol $\bar{S}(x)$; x_{i+1} is said to be generated by x_i. The possible digitadition series are separated into three types: A — each x_i prime to 3; B — each x_i divisible by 3 but not by 9; C — each x_i divisible by 9. Given x it may or may not be possible to find a y that generates x. If no y exists, x is called a self number. For example, the numbers 1, 3, 5, 7, 9, 20, 31, 42, 53 ,64, 75, 86, 97 are the self numbers less than 100. It may, on the other hand, happen that x is generated by several different numbers $y, z, \cdots$; such an x is called a junction number. It is stated that if x and y are of the same type (that is, each prime to 3, or each divisible by 3 but not 9, or each divisible by 9) then $\bar{S}(x)$ and $\bar{S}(y)$ coincide after a certain point.

The booklet contains a great deal of numerical data. In particular it contains a list of the self numbers from 1 to 1313, also a list of junction numbers from 101 to 1015. It is noted that pairs of self numbers differing by 2 exist.

In conclusion the author gives a rather complicated criterion for a self number. He also notes the conjecture that a digitadition series cannot contain more than 4 consecutive primes. Finally it is remarked that numbers exist with more than 2 generators.

L. Carlitz (Durham, N.C.)

6382:

McCarthy, Paul J. On a certain family of arithmetic functions. Amer. Math. Monthly **65** (1958), 586–590.

Let r be a positive integer. An integer is said to be r-free if it is not divisible by the rth power of any prime, and the function $T_r(n)$ is defined to be the number of integers k such that $1\leqq k\leqq n$ and the greatest common divisor (k, n) is r-free. It is shown that $T_r(n)$ is multiplicative in the usual number-theoretic sense, and hence it is given explicitly by $T_r(n)=n\prod(1-p^{-r})$, where p ranges over all primes such that $p^r|n$. For $r>1$ it is shown that

$$\sum_{m=1}^{n} T_r(m)=\frac{n^2}{2\zeta(2r)}+O(n),$$

where $\zeta(s)$ is the Riemann zeta function. From this it follows that the probability that the greatest common divisor of two positive integers be r-free is $1/\zeta(2r)$.

W. H. Mills (New Haven, Conn.)

6383:

Alder, Henry L. A generalization of the Euler ϕ-function. Amer. Math. Monthly **65** (1958), 690–692.

Let m be a non-negative integer and n a positive integer. The function $\varphi(n, m)$ is defined as the number of ordered pairs $\langle x, y\rangle$ for which $x+y=n+m$, $1\leqq x\leqq n$, and x and y are both relatively prime to n. The function $\varphi(n, m)$ is multiplicative (as a function of n) and hence,

$$\varphi(n, m)=n\prod\left(1-\frac{e_m(p)}{p}\right),$$

where p ranges over the prime divisors of n, $e_m(p)=1$ if $p|m$, and $e_m(p)=2$ if $p\nmid m$. For $m=0$ this becomes the usual formula for the Euler phi-function, and for $m=1$ we obtain a result of Schemmel [J. Math. **70** (1869), 191–192].

W. H. Mills (New Haven, Conn.)

6384:

Carlitz, L. Some congruences involving binomial coefficients. Elem. Math. **14** (1959), 11–13.

Extending results of Glaisher [Quart. J. Math. **31** (1900), 1–35], the author derives a number of congruence properties of the sum

$$\Delta_r=\sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}\binom{nw+sw-1}{p-1},$$

where n is an arbitrary integer, p is a prime >3 and $p^e|w$, $e\geqq 1$. For example, he proves that, for $0<2t<p-1$,

$$\Delta_{2t}\equiv-\frac{(2t)!}{2t+1}B_{p-2t-1}w^{2t}p \pmod{p^{2te+2}},$$

where B_m denotes the mth Bernoulli number in the even suffix notation. He also obtains similar congruences for odd values of r.

A. L. Whiteman (Los Angeles, Calif.)

6385:

Jarden, Dov. 'Genuine' composite numbers. Riveon Lematematika **12** (1958), 36. (Hebrew)

Following Hardy and Wright, a "genuine composite number" is a number which is known to be composite without any proper factor being known. The author refutes the (outdated) statements by Hardy and Wright [*An introduction to the theory of numbers*, 3rd ed., Clarendon Press, Oxford, 1954; MR **16**, 673; p. 16] and B. M. Stewart to the effect that $M_{257}=2^{257}-1$ is the "largest genuine composite number" by observing that: if n is a genuine composite number, then so are 2^n-1 and U_n (the nth Fibonacci number); and hence there can be no "largest genuine composite number."

E. G. Straus (Los Angeles, Calif.)

6386:

Schinzel, A. Sur un problème de P. Erdös. Colloq. Math. **5** (1958), 198–204.

If k and n are positive integers, then the product $n(n-1)\cdots(n-k+1)$ is divisible by $k!$. P. Erdös raised the question of whether there is always one factor which can be omitted without violating the divisibility by $k!$, i.e., whether there exists an i $(0\leqq i<k)$ such that $(n-i)k!$ divides $n(n-1)\cdots(n-k+1)$. If this holds, we say that $H_{k,n}$ is true. If k is such that $H_{k,n}$ is true for all n, then we say that H_k is true.

The author shows that H_k is false if $k=15$, 21, 22, 33, but true for all other values of $k\leqq 33$. H_k is true if k is a power of a prime, and the author conjectures that H_k is false in all other cases, with at most a finite number of exceptions. The paper ends with a proof, by P. Erdös, that H_k is false for infinitely many k.

N. G. de Bruijn (Amsterdam)

6387:

✶Гельфонд, А. О. Решение уравнений в целых числах. 2. Изд. [Gel'fond, A. O. Solution of equations in integers. 2nd ed.] Populyarnye Lekcii po Matematike, Vypusk 8. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1957. 63 pp. 0.85 rubles.

An essentially unchanged republication of the booklet listed in MR **13**, 913.

6388:

Bombieri, Enrico. Sulle soluzioni intere dell'equazione $4x^3=27y^2+N$. Riv. Mat. Univ. Parma **8** (1957), 199–206.

All integral solutions (x, y) of the title equation are determined for $-22\leqq N\leqq 80$ except $N=49$. The argument turns on the fact that, for the values of N involved, the cubic t^3-xt+y in the indeterminate t must have a rational integral root (for otherwise the roots would generate a cubic field, which contradicts what is known about the possible values of the discriminants of cubic fields.

J. W. S. Cassels (Cambridge, England)

3689:

Moessner, Alfred; and Xeroudakes, George. On the solutions of the system $2A^m=B^m+C^m+(B+C)^m$ ($m=2,4$). Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske. Ser. II. 13 (1958), 89–96. (Serbo-Croatian summary)

Various identities, involving equal sums of like powers, are established. Numerical illustrations are given.

W. H. Mills (New Haven, Conn.)

6390:

Lánský, Miloš. On prime lattice points lying on the conics. Mat.-Fyz. Časopis. Slovensk. Akad. Vied 7 (1957), 121–127. (Czech. Russian and English summaries)

Es sei r eine rationale Zahl. Gibt es einen nicht zerfallenden Kegelschnitt, der durch den Punkt $[0, r]$ hindurchgeht und unendlich viele Punkte $[x, y]$ besitzt, wobei x eine Primzahl und y eine ganze Zahl ist, so stellt er eine Parabel dar, deren Achse zu der y-Achse parallel ist. Aus dem Satz werden einige Folgerungen abgeleitet, u.a.: Es gibt keinen nicht zerfallenden Kegelschnitt, der die Koordinatenachsen in Punkten mit rationalen Koordinaten schneidet und auf dem unendlich viele Punkte liegen, deren beide Koordinaten Primzahlen sind.

M. Novotný (Brno)

6391:

Stolt, B. A theorem on triangular numbers. Portugal. Math. 16 (1957), 3–5.

Let Δ_k be the kth triangular number, $k(k+1)/2$. For a given positive integer n, let k be determined by $\Delta_k\leq n<\Delta_{k+1}$. It is shown that n can be represented in the form

$$n=\Delta_x+\Delta_y+\Delta_u+\Delta_v,$$

where $x+y+u+v=2k-1$, thus establishing a theorem first stated without proof by F. Pollock in 1854.

L. Moser (Edmonton, Alta.)

6392:

Ugrin-Šparac, Dimitrije. Some properties of ternary cubic forms $x^3+my^3+m^2z^3-3mxyz$. Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske Ser. II. 12 (1957), 23–29. (Serbo-Croatian summary)

Let $F_m(x, y, z)=x^3+my^3+m^2z^3-3mxyz$. It is shown that every positive prime $p\neq 3$ can be represented by F_1. If $p\equiv -1 \pmod 3$ the representation is unique and if $p\equiv 1 \pmod 3$ there are two representations, disregarding distinction between x, y and z. If $m\not\equiv\pm 1 \pmod 7$, the integers $7^n a$ with $n\not\equiv 0 \pmod 3$ and $(a, 7)=1$ are not representable by F_m. Some further results on the form F_m lead to a proof that the field $k(2^{\frac{1}{3}})$ is Euclidean.

L. Moser (Edmonton, Alta.)

6393:

Korobov, N. M. Estimation of rational trigonometrical sums. Dokl. Akad. Nauk SSSR (N.S.) 118 (1958), 231–232. (Russian)

The author announces a new estimate for exponential sums of the form

$$S=\sum_{x=1}^{P} e((a_{n+1}x^{n+1}+\cdots+a_1x)/q),\quad e(\theta)=e^{2\pi i\theta},$$

where $a_{n+1}, \cdots, a_1, q$ are integers and $(a_{n+1}, q)=1$. The improvement relates to the dependence on n, which is important for certain applications (see next review). Put $P=q^{1/r}$, and suppose $1<r<n+1$. Suppose also that the least prime divisor of q is greater than $n+1$. Then the new estimate is

$$|S|\leq CP^{1-\rho},\quad \rho=\frac{\alpha r(n+1-r)}{n^4l^2},$$

where C and α are absolute constants and $l=\log(2n/(n+1-r))$. The proof is said to make essential use of the fact that the polynomial has rational coefficients, and to depend on the following auxiliary result. Let n, r, k, τ, q, P be integers satisfying

$$1\leq r\leq n,\quad \tau\geq 1,\quad k>n(n+\tau),\quad q>2k,$$
$$\left(\frac{q}{2k}\right)^{1/(r+1)}<P\leq\left(\frac{q}{2k}\right)^{1/r}.$$

Then the number of solutions of the n simultaneous congruences

$$x_1^s+\cdots+x_k^s\equiv y_1^s+\cdots+y_k^s \pmod q,\quad 1\leq s\leq n,$$

with $x_1, \cdots, y_k$ between 1 and P, is less than

$$\exp(c(n+\tau)k\log k)P^{2k-\nu},$$
$$\nu=\tfrac{1}{2}r(2n+1-r)-\tfrac{1}{2}r(n+1)(1-1/n)^\tau.$$

This is said to be proved by the methods of Vinogradov. The estimate is nearly best possible if τ is large.

H. Davenport (Cambridge, England)

6394:

Korobov, N. M. On zeros of the $\zeta(s)$ function. Dokl. Akad. Nauk SSSR (N.S.) 118 (1958), 431–432. (Russian)

The author announces a further improvement in his estimate for exponential sums containing a polynomial with rational coefficients. With the notation of the preceding review, and with the same assumption about the least prime factor of q, suppose now that $n+\varepsilon\leq r\leq n+1-\varepsilon$, where $\varepsilon>0$. Then there exist an absolute constant C and a constant α depending only on ε, such that

$$|S|\leq CP^{1-\sigma},\quad \sigma=\frac{\alpha}{(n\log n)^{5/2}}.$$

This estimate is said to lead to the result

$$\zeta(1+it)=O(\log t)^{5/7+\varepsilon}$$

as $t\to\infty$, and (as a straightforward consequence)

$$\pi(x)=\mathrm{li}\,x+O(x\exp(-a(\log x)^{7/12-\varepsilon})),$$

where a depends only on ε. The details will be awaited with interest.

H. Davenport (Cambridge, England)

6395:

Korobov, N. M. New number-theoretic estimates. Dokl. Akad. Nauk SSSR (N.S.) 119 (1958), 433–434. (Russian)

In a companion paper [#6394 above] the author proved the following theorem: For $n>1$, let $f(x)=\alpha_1x+\cdots+\alpha_{n+1}x^{n+1}$, where each α_ν is real. Let P and Q be positive integers, $P>1$, and define r by the equation $P^r=|\alpha_{n+1}^{-1}|$. If for every fixed δ satisfying $0<\delta<\frac{1}{2}$ we have $n+\delta\leq r\leq n+1-\delta$, then there exist an absolute constant C and another constant $\alpha=\alpha(\delta)$ such that

$$\left|\sum_{x=Q+1}^{Q+P} e^{2\pi if(x)}\right|<CP^{1-\alpha/(n\log n)^{2/3}}.$$

This is an improvement of similar estimates due to Vinogradov and Hua, and leads to corresponding improvements of estimates in various problems of analytic number theory. Improved results for $\zeta(1+it)$ and $\pi(x)-\mathrm{li}\,(x)$ are discussed in the paper above. In this paper he describes other improvements, for example

$$\sum_{n\leq x}\sigma(n)=\frac{\pi^2}{12}x^2+O(x\{\log x\}^{5/7+\varepsilon})$$
$$\sum_{n\leq x}\phi(n)=\frac{3}{\pi^2}x^2+O(x\{\log x\}^{5/7+\varepsilon}).$$

T. M. Apostol (Pasadena, Calif.)

6396:

Meller, N. A. Computations connected with the check of Riemann's hypothesis. Dokl. Akad. Nauk SSSR 123 (1958), 246–248. (Russian)

6397:

Locher-Ernst, L. Bemerkungen über die Verteilung der Primzahlen. Elem. Math. 14 (1959), 1–5.

Various forms of the prime number theorem are stated, and it is observed that an excellent approximation for $\pi(n)$ for $50<n<2000$ is given by $n(1/3+1/4+\cdots+1/n)^{-1}$. Euler's recurrence for the sum of divisors of n is explained but not proved. *L. Moser* (Edmonton, Alta.)

6398:

Corrádi, Keresztély. Über die Zusammenhang der Primzahlsätze arithmetischer Progressionen desselben Differenzes. Mat. Lapok 9 (1958), 67–90. (Hungarian. Russian and German summaries)

The author proves by elementary methods the following result: Let k be any integer. If we assume that the prime number theorem for arithmetic progressions holds for one progression $l \pmod k$, $(l, k)=1$, then it holds for every such progression.

The author does not use Selberg's formula.

P. Erdős (Birmingham)

6399:

Kanold, Hans-Joachim. Über quadratfreie Zahlen mit vorgeschriebener Primteileranzahl. Arch. Math. 9 (1958), 46–53.

Let $A_k(x)$ denote the number of integers $m\leq x$ which may be represented as the product of k distinct primes $p_1<\cdots<p_k$. It is well known that $A_k(x)\sim[x/\log x]\cdot[(\log\log x)^{k-1}/(k-1)!]$. The author proves that the same result holds if certain conditions are imposed: either (1) $p_1>\exp(\log x)^{\varepsilon(x)}$, $p_k>x^{1-\log\log x}$, where $\varepsilon(x)$ is any function tending to 0 as $x\to\infty$; or (2) $(\sigma_r(m), m)=(\varphi_r(m), m)=1$, where $\varphi_r(m)=m^r\prod_{p|m}(1-p^{-r})$, $\sigma_r(m)=\sum_{d|m}d^r$. The proofs are straightforward, using known results on the distribution of primes, and the results are not entirely unexpected. *R. D. James* (Vancouver, B.C.)

6400:

Linnik, Yu. V. Solution of some binary additive problems by computing dispersion in progressions. Dokl. Akad. Nauk SSSR 123 (1958), 975–977. (Russian)

The equation $n=p_1p_2+\zeta^2+\eta^2$ is considered; p_1, p_2 primes; $p_2\leq n^\alpha$; α a small constant; $\zeta^2+\eta^2$ squarefree. The proof of an asymptotic formula for the solution quantity with a sufficiently good remainder term is sketched. Some other binary problems are considered.

Summary provided by the author

6401:

Linnik, Yu. V. Hardy-Littlewood problem on representation as the sum of a prime and two squares. Dokl. Akad. Nauk SSSR 124 (1959), 29–30. (Russian)

The author claims to prove the Hardy-Littlewood heuristic asymptotic formula for the equation $n=p+\zeta^2+\eta^2$. Yet owing to a gap in his proof, his arguments lead to the weaker result that the Hardy-Littlewood equation is solvable for all sufficiently large n and the representation quantity satisfies the inequality:

$$a(n)>0.7\pi\ \mathrm{Li}(n)\prod_p\left(1+\frac{\chi_4(p)}{p(p-1)}\right)\prod_{p|n}\frac{(p-1)(p-\chi_4(p))}{p^2-p+\chi_4(p)}.$$

Some other binary problems are considered.

Summary provided by the author

6402:

Duncan, R. L. A topology for sequences of integers. Amer. Math. Monthly 66 (1959), 34–39.

Let $A=(a_n)$ be a strictly increasing sequence of positive integers, and $A(n)$ be the number of elements of A not exceeding n. The author topologizes the set of all such sequences A for which $d(A)=\mathrm{Lim}_n\, n^{-1}A(n)$ exists. The topology is given by the metric $D(A,B)=0$ if $a_n=b_n$ for all n, and $D(A,B)=k^{-1}+|d(A)-d(B)|$, where k is the least integer n for which $a_n\neq b_n$. Topological properties of this metric space are investigated.

M. Brown (Ann Arbor, Mich.)

6403:

Turán, Pál. On the distribution of "digits" in Cantor-systems. Mat. Lapok 7 (1956), 71–76. (Hungarian. Russian and English summaries)

The author proves, by using the Cantor-Lebesgue theorem, the following theorem. Let $0\leq x\leq 1$,

$$x=\sum_{r=1}^{\infty}\frac{c_r}{q_1q_2\cdots q_r},$$

$q_r\geq 2$, $0\leq c_r\leq q_r-1$, with c_r, q_1, q_2, $\cdots$ integers. Assume $\sum_{r=1}^{\infty}1/q_r<\infty$. Then for almost all x

$$\sum_{r=1}^{\infty}\frac{\min(c_r,\,(q_r-c_r),\,|c_r-q_r/2|)}{q_r}$$

diverges. The author remarks that one can obtain stronger results by using probability methods.

P. Erdős (Birmingham)

6404:

Rényi, Alfréd. On the distribution of the digits in Cantor's series. Mat. Lapok 7 (1956), 77–100. (Hungarian. Russian and English summaries)

In this paper the author proves that the well-known theorem of Borel on the frequency of the digits of the q-ary expansions can be generalised for Cantor series if $\sum 1/q_n=\infty$. In a previous paper [Acta Math. Acad. Sci. Hungar. 6 (1955), 285–335; MR 18, 339] the author proved a slightly weaker theorem, but the present proof is simpler. The case $\sum_{n=1}^{\infty}1/q_n<\infty$ is also considered.

[Cf. #6406, #6407 below]. *P. Erdős* (Birmingham)

6405:

Marczewski, E. Remarks on the Cantor-expansions of real numbers. Mat. Lapok 7 (1956), 212–213. (Hungarian)

The author observes that the result of Turán [#6403] follows from the law of large numbers.

P. Erdős (Birmingham)

6406:

Szüsz, Péter. Bemerkungen zur Verteilung der Ziffern in der Cantorschen Reihe reeller Zahlen. Mat. Lapok 8 (1957), 68–78. (Hungarian. German and Russian summaries)

Hungarian version of #6407 below.

6407:

Szüsz, P. Bemerkung über die Verteilung der Ziffern in der Cantorschen Reihe reeller Zahlen. Acta Math. Acad. Sci. Hungar. 8 (1957), 163–168.

Let $q_1, q_2, \cdots$ $(q_n\geq 2)$ be an infinite sequence of integers. It is well known that for every x $(0<x<1)$ we have

$$x=\sum_{k=1}^{\infty}\frac{\varepsilon_k(x)}{q_1q_2\cdots q_k},\quad 0\leq\varepsilon_k(x)\leq q_k-1\quad(\varepsilon_k(x)\text{ an integer}).$$

Denote by $N_n(x,r)$ the number of solutions of $\varepsilon_k(x)=r$, $1\leq k\leq n$. Rényi proved [#6404 above] that if $\sum_{k=1}^{\infty}1/q_k=\infty$

then, if $q_k > \max(r, s)$ for all $k > k_0$, we have for almost all x

(*) $\lim_{n\to\infty} N_n(x, r)/N_n(x, s) = 1.$

But he also proved that to every x $(0<x<1)$ there exists a sequence q_k satisfying $\sum_{k=1}^{\infty} 1/q_k = \infty$, $q_n \to \infty$, which does not satisfy (*).

The author sharpens and deepens this last result of Rényi. He proves that for every x $(0<x<1)$ and every sequence $c_1, c_2, \cdots$ $(c_k \geq 2)$ of integers there exists a sequence of integers $q_1, q_2, \cdots$ satisfying $0 \leq q_k - c_k < 16$ and $1 \leq \varepsilon_k(x) \leq q_k - 1$ (i.e., $\varepsilon_k(x)$ is never 0).

P. Erdős (Birmingham)

6408:

Rieger, G. J. Über die Anzahl der Ideale in einer Idealklasse mod $\mathfrak{f}$ eines algebraischen Zahlkörpers. Math. Ann. **135** (1958), 444–466.

Let K be an algebraic number field of degree n and discriminant Δ, and let r_1 of the conjugate fields be real and $2r_2$ complex $(r_1+2r_2=n)$. Let $\mathfrak{f}$ be an ideal in K and let $\mathfrak{H}$ be an ideal class (mod $\mathfrak{f}$) in the narrow sense. Let $H(x; \mathfrak{H})$ denote the number of ideals in $\mathfrak{H}$ with norm not exceeding x. It has long been known, from consideration of the number of integer points in a large region, that $H(x; \mathfrak{H})$ is approximately

$$\frac{(2\pi)^{r_2} R(\mathfrak{f}) x}{w(\mathfrak{f}) |\Delta|^{\frac{1}{2}} N(\mathfrak{f})},$$

where $w(\mathfrak{f})$ is the number of totally positive roots of unity $\equiv 1 \pmod{\mathfrak{f}}$ and $R(\mathfrak{f})$ is the regulator of a generating system of totally positive units $\equiv 1 \pmod{\mathfrak{f}}$. An asymptotic formula as $x\to\infty$ for fixed $\mathfrak{f}$ was given by Landau [Math. Z. 2 (1918), 52–154], the error term being $O(x^{1-2/(n+1)})$. The result of this paper is that $H(x; \mathfrak{H})$ differs from the above expression by less than

$$c(K) R(\mathfrak{f}) (x N(\mathfrak{f}))^{1-1/n},$$

where $c(K)$ depends only on K. The proof is essentially elementary, though it is remarked that a result of a similar character could be proved analytically. Some applications are to be given in a later paper.

H. Davenport (Cambridge, England)

6409:

Fáy, Árpád. On Markoff's numbers. Mat. Lapok 7 (1956), 262–270. (Hungarian. Russian and English summaries)

The author defines a set M of real numbers as follows: $Z \in M$ if and only if there exist homogeneous linear forms $\xi(x, y)$, $\eta(x, y)$ with real coefficients for which $z = m/|\Delta|$, where Δ is the determinant of $\xi\eta$ and $m = \inf |\xi\eta|$ over all pairs of integers $(x, y) \neq (0, 0)$. Markoff determined all the points of the set M in $(1/\eta, \infty)$. The author gives new proofs for some of these results of Markoff and also shows that M is closed and has no points in the open interval $(1/2\sqrt{3}, 1/13)$. {This last result was known and is due to Shibata [Koksma, Diophantische Approximationen, Springer, Berlin, 1936; p. 33].}

P. Erdős (Birmingham)

6410:

Kogoniya, P. G. Condensation points of the set of Markov numbers. Dokl. Akad. Nauk SSSR (N.S.) **118** (1958), 632–635. (Russian)

Let $M(N)$ be the set of all irrational numbers α $(0<\alpha<1)$ whose continued fraction partial quotients a_k satisfy $\limsup_{k\to\infty} a_k = N$, where $N=1, 2, 3, \cdots$. Let $L(\alpha)$ be the lower bound of the set of all positive numbers c for which

$$\left|\alpha - \frac{p}{q}\right| < \frac{c}{q^2}$$

has infinitely many solutions in integers p, $q>0$, and let $M_L(N)$ be the set of all values $L(\alpha)$ for $\alpha \in M(N)$. It is shown that $(N^2+4N)^{-\frac{1}{2}}$, which is the smallest number in $M_L(N)$, is a point of condensation of $M_L(N)$ $(N\geq 2)$ and that $22/(65+9\sqrt{3})$ is the greatest point of condensation of $M_L(3)$. [For earlier work of the author on related problems see Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **19** (1953), 121–133; MR **16**, 451.]

R. A. Rankin (Glasgow)

6411:

Polosuev, A. M. On a problem concerned with a uniform distribution of a system of functions. Dokl. Akad. Nauk SSSR **122** (1958), 346–348. (Russian)

Let $f_1(x), f_2(x), \cdots, f_s(x)$ be polynomials with integral coefficients and not identically zero. A method of Korobov [Izv. Akad. Nauk SSSR. Ser. Mat. **17** (1953), 389–400; MR **15**, 511] is generalized to show that, for certain λ_i, α_i $(1\leq i\leq s)$, the system of functions

$$\alpha_1\lambda_1^x f_1(x), \cdots, \alpha_s\lambda_s^x f_s(x)$$

is uniformly distributed in s-dimensional space.

R. A. Rankin (Glasgow)

6412:

Krupička, Svatopluk. Über die Anzahl der Gitterpunkte in mehrdimensionalen konvexen Körpern. Czechoslovak Math. J. **7(82)** (1957), 524–552. (Russian. German summary)

For the number of lattice points in r-dimensional ellipsoids the formulas $A(x) = x^{r/2}V + P(x)$ are well known, where V denotes the volume of the "fundamental" ellipsoid. If x tends to infinity, then for the remainder $P(x)$ we have the estimate from above $P(x) = O(x^{(r/2)-r/(r+1)})$ and the estimate from below $P(x) = \Omega(x^{(r-1)/4})$. In the present article the author proves the correctness of analogous formulas for the number of lattice points in more general convex r-dimensional bodies.

Author's summary

COMMUTATIVE RINGS AND ALGEBRAS

See also 6425, 6434, 6466.

6413:

Fried, Ervin. Algebraically closed fields as finite extensions. Mat. Lapok 7 (1956), 47–60. (Hungarian. Russian and English summaries)

Let A be an algebraically closed field and $V \subset A$ a proper subfield of A. Suppose that A is obtained from V by adjunction of a finite number of (a priori not necessarily algebraic) elements. Then V is a real closed field and $A = V(i)$. The proofs and formulations are modifications and slight improvements of known results. [See N. Bourbaki, Algèbre, chap. VI, Actualités Sci. Ind. no. 1179, Hermann, Paris, 1952; MR **14**, 237; problems on pp. 47–48.]

Further it is shown: If F is a field and all irreducible polynomials over F have a degree less then a fixed integer, then either F is algebraically closed (and all irreducible polynomials are of degree ≤ 1) or F is real closed (and all irreducible polynomials are of degree ≤ 2).

Št. Schwarz (Bratislava)

6414:

Auslander, Maurice; and Buchsbaum, David A. Codimension and multiplicity. Ann. of Math. (2) **68** (1958), 625–657.

This is an expository paper on the theory of multi-

plicity and is based on the Koszul complex. Work on this method was done first by Serre in 1955 (mimeographed notes): multiplicities of an ideal with respect to a module can be expressed as the alternative sum of length of homologies of a certain complex derived from a Koszul complex; this led to a theory of multiplicity.

The main objects of this paper are (1) to generalize the results of Serre referred to above, (2) to give an axiomatic description of multiplicity, and (3) to give a unified treatment of Macauley modules [cf. Nagata, Proc. Internat. Sympos. Algebraic Number Theory, Tokyo-Nikko, 1955, pp. 191–226; Science Council of Japan, Tokyo, 1956; MR 18, 637; Northcott, Mathematica 3 (1956), 117–126; MR 18, 637; Rees, Proc. Cambridge Philos. Soc. 53 (1957), 28–42; MR 18, 637].
M. Nagata (Cambridge, Mass.)

6415:

Northcott, D. G. Dilatation properties of regular local rings. Proc. Cambridge Philos. Soc. **55** (1959), 1–9.

Soient Q un anneau local régulier de dimension d, $\mathfrak{m}$ son idéal maximal, $G(Q)$ l'anneau gradué associé $\sum_{n=0}^{\infty} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ (qui est un anneau de polynômes à d variables sur $K=A/\mathfrak{m}$), P un point (algébrique sur K) de l'espace projectif de dimension $d-1$ sur K et $\mathfrak{J}$ l'idéal homogène de $G(Q)$ correspondant à P; pour $x \in Q$, notons $v(x)$ le plus grand entier n tel que $x \in \mathfrak{m}^n$ (v est une valuation de Q) et $G(x)$ la classe de x dans $\mathfrak{m}^{v(x)}/\mathfrak{m}^{v(x)+1}$. L'ensemble Q_P des quotients x/y, où $x, y \in Q$, $v(x) \geqq v(y)$ et $G(y) \notin \mathfrak{J}$, est un sous-anneau local régulier du corps des fractions de Q; on dit que Q_P est un anneau du premier voisinage de Q. L'idéal $\mathfrak{m}Q_P$ est principal, et $Q_P/\mathfrak{m}Q_P$ est un anneau local régulier de dimension $d-1$ qui s'identifie canoniquement à l'anneau local de P dans l'espace projectif de dimension $d-1$ sur K. On appelle suite de résolution d'origine Q une suite $Q=Q_0, Q_1, \cdots, Q_s$ d'anneaux locaux réguliers tels que, pour tout j, Q_j soit un anneau du premier voisinage de Q_{j-1}; posons $Q_j\mathfrak{m}_{j-1}=Q_jx_j$, où $x_j \in Q_j$; pour $p<r$, on dit que Q_r est proche de Q_p si $x_{p+1}/(x_{p+2}x_{p+3}\cdots x_r)$ est élément de l'idéal maximal de Q_r. On montre que Q_r est proche d'au plus d de ses prédécesseurs. En utilisant ceci et le fait que tout anneau local complet est quotient d'un anneau local régulier, on obtient le résultat suivant: soient A un anneau local de dimension 1 dont l'idéal maximal $\mathfrak{p}$ ne se compose pas uniquement de diviseurs de zéro, et d l'entier $[(\mathfrak{p}/\mathfrak{p}^2):(A/\mathfrak{p})]$; on pose $d'=d$ dans le cas d'égales caractéristiques et $d'=d+1$ dans celui d'inégales caractéristiques; si $A, A_1, A_2, \cdots$ est une branche d'origine A, alors chaque A_j est proche d'au plus d' de ses prédécesseurs [terminologie de l'auteur, Proc. London Math. Soc. (3) 8 (1956), 388–415; MR 17, 938]. *P. Samuel* (Clermont-Ferrand)

ALGEBRAIC GEOMETRY

See also 6415, 6721.

6416:

Wu, Ding-jar. The investigation of the singularity $S_{1,m-4}{}^m(P)$ of a plane curve. Advancement in Math. 3 (1957), 445–451. (Chinese. English summary)

6417:

Anisimova, E. P. A construction in the space of collineations which maps certain ruled surfaces on themselves. Moskov. Oblast. Pedagog. Inst. Uč. Zap. 57 (1957), 153–156. (Russian)

L'A. donne la construction géométrique des homographies qui reproduisent une surface cubique d'un espace S_3. *A. Švec* (Prague)

6418:

Manevič, V. A. A 6th degree complex of straight lines generated by a tetrahedral complex. Dokl. Akad. Nauk SSSR 122 (1958), 183–185. (Russian)

L'A. étudie les propiétés d'un complexe de droites Σ^6 d'ordre 6 qui est engendré par un complexe quadratique tétraèdral Π^2. *A. Švec* (Prague)

6419:

Adamo, Marco. Alcune proprietà della $V_3{}^4$ di S_5 con sei punti doppi indipendenti, e sua transformazione nella $V_3{}^3$ di C. Segre. Rend. Sem. Fac. Sci. Univ. Cagliari 27 (1957), 169–180.

6420:

Gallarati, Dionisio. Alcune osservazioni sulle irregolarità di un S_3 doppio. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 24 (1958), 139–142.

Si provano alcune proprietà delle varietà algebriche irregolari ($g_1>0$) a tre dimensioni, V_3, rappresentabili doppiamente sopra uno spazio proiettivo $P_3(C)$. Un teorema del De-Franchis assicura che le superficie di diramazione su $P_3(C)$ è spezzata in $2g_1+2$ superficie di un fascio. Supponendo verificate alcune ipotesi di generalità riguardo a tale fascio, si dimostra qui che per le V_3 in esame è $g_2=0$. Tenendo conto che, per ipotesi, $g_1>0$, si riconosce che tali V_3 posseggono un fascio irrazionale di superficie algebriche. *F. Gherardelli* (Florence)

6421:

Godeaux, Lucien. Sur les systèmes invariants de certaines surfaces algébriques. J. Math. Pures Appl. (9) 37 (1958), 161–172.

The author first defines a surface F of order $3\nu+1$ in S_3, having on it a cyclic involution of order $9\nu^2+3\nu+1$ defined by a projective self-transformation of this period. Subsequently, attention is confined to the case $\nu=2$, when F has the equation

$$a_1x_1{}^7x_2+a_2x_2{}^7x_3+a_3x_3{}^7x_1+a_4x_4{}^8+a_5x_1x_2x_3x_4{}^5+a_5x_1{}^2x_2{}^2x_3{}^2x_4{}^2=0$$

and the transformation is

$$x_1:x_2:x_3:x_4 \to x_1:\varepsilon x_2:\varepsilon^{37}x_3:\varepsilon^{27}x_4,$$

ε being a primitive 43rd root of unity. The vertices X_1, X_2, X_3 of the tetrahedron of reference are the only united points of the involution.

A model Φ of the involution is constructed, whose hyperplane sections include the images of sets of 43 conjugate plane sections of F; Φ is of order 344 in S_{158}, and has section genus 189. The singularities X_1', X_2', X_3' of Φ corresponding to the united points X_1, X_2, X_3 are analysed by the author's usual method; the first neighbourhood of each consists of a conic and two lines intersecting consecutively in a simple point and a binode B_7.

Attention is then turned to the canonical and bicanonical systems on Φ, which are the images of the systems traced on F by the linear systems of surfaces

$$x_4(x_1x_2x_3+\lambda x_4{}^3)=0,$$
$$\lambda_1x_1{}^7x_2+\lambda_2x_2{}^7x_3+\lambda_3x_3{}^7x_1+\lambda_4x_4{}^8+\lambda_5x_1x_2x_3x_4{}^5=0,$$

respectively (both these are compounded with the involution). In particular, the image K_1' of the section K_1

of F by $x_4=0$ is a fixed part of the canonical system, the residual part being a pencil of elliptic curves, with no base points except X_1', X_2', X_3', one member of which is $3K_1'$. K_1' is rational and of grade -1, and meets the variable bicanonical curve in one variable point.

On these grounds the author asserts that K_1' is not an exceptional curve on Φ, in spite of being rational and of grade -1. This of course is a question of definition. It does not appear to mean that K_1' cannot be transformed into the neighbourhood of a simple point on a birational image of Φ. The bicanonical model is in fact an octavic surface in S_3, on which K_1' appears as a line, containing three singular points, the images of X_1, X_2, X_3; and it seems to the reviewer that, on the transform of this surface by cubics through these three points, K_1' appears as the neighbourhood of a simple point.

P. Du Val (London)

6422:

Godeaux, Lucien. Sur les surfaces de genres arithmétique et géométrique nuls possédant une courbe bicanonique effective. Acad. Roy. Belg. Bull. Cl. Sci. (5) **44** (1958), 809–812.

F is a non-singular regular surface of genera $p_g=p_a=2$, whose canonical system is a pencil of irreducible curves of genus $p^{(1)}$. F has a birational self transformation of period 3, without fixed points, base points, or exceptional curves. The image of the induced cubic involution on F is a regular surface F', with genera $p_g=p_a=0$, linear genus π given by

$$3(\pi-1)=p^{(1)}-1$$

and plurigenera $P_2=P_4=1$, $P_3=P_5=2$, $P_6=3$. The tricanonical system is a pencil $|K|$ each curve of which is the image of three curves (permuted cyclically by the transformation) of the canonical pencil on F. Two curves of this are of the form $3K_1$, $3K_2$; the canonical system is

$$|2K_1-K_2|=|2K_2-K_1|$$

and the unique bicanonical curve is K_1+K_2.

P. Du Val (London)

6423:

Burau, Werner. Zur Geometrie der verallgemeinerten Raumelemente des P_n und der zugehörigen J-Mannigfaltigkeiten. Abh. Math. Sem. Univ. Hamburg 22 (1958), 141–157.

P_n is n-dimensional projective space. $G_{n,k}$ is the Grassmannian of k-dimensional linear subspaces X_k of P_n. $G_{n,k}^{(h)}$ is the Veronesian of order h of $G_{n,k}$, i.e., projective model of its sections by all hypersurfaces of order h. $\Pi_{n,n_1,\cdots,n_s}^{h_1,\cdots,h_s}$ is the Segre product of $G_{n,n_1}^{(h_1)}, \cdots, G_{n,n_s}^{(h_s)}$, where $n>n_1>n_2>\cdots>n_s\geqq 0$, and its points are thus in correspondence with sets of subspaces $X_{n_1}, \cdots, X_{n_s}$ in P_n. On this is a variety $J_{n,n_1,\cdots,n_s}^{h_1,\cdots,h_s}$ (actually a linear section) representing sets such that

$$X_{n_1}\supset X_{n_2}\supset\cdots\supset X_{n_s}.$$

(Such a set is a generalised space element.)

Every projective self-transformation of P_n induces a projective self-transformation of the ambients of $\Pi_{n,n_1,\cdots,n_s}^{h_1,\cdots,h_s}$ and of $J_{n,n_1,\cdots,n_s}^{h_1,\cdots,h_s}$. This is the most general irreducible representation of the projective group in P_n. The paper is devoted to a detailed study of this group, especially of its invariant spaces, regarded as ambients of configurations forming part of or simply related to $\Pi_{n,n_1,\cdots,n_s}^{h_1,\cdots,h_s}$ and $J_{n,n_1,\cdots,n_s}^{h_1,\cdots,h_s}$.

P. Du Val (London)

6424:

Segre, Beniamino. Plans graphiques algébriques réels non desarguésiens et correspondances crémoniennes topologiques. Rev. Math. Pures Appl. 1 (1956), no. 3, 35–50.

A "plan graphique" (projective plane) is a set π of "points" and a collection L of subsets, "lines", of π such that (i) there is a unique line which contains an arbitrary pair of points; (ii) two distinct lines have a unique common point; and (iii) each line contains at least three points. If π is the real projective plane and the collection L is an algebraic system of real algebraic curves of π, the plan graphique is algebraic. The existence of a nondesarguesian algebraic plan graphique has been an open question. In this paper the author settles this question by constructing one such and lays the foundations for a theory of real topological cremona transformations.

It is shown that an algebraic plan graphique is nondesarguesian if the system L of algebraic curves contains a curve of positive genus. A nondesarguesian plan graphique is constructed in which the collection L is a rational system containing real elliptic cubics.

A real cremona transformation between two real projective planes is said to be topological if it is one-to-one. A real algebraic curve is said to be pure [impure] if it does not [does] possess a real isolated point, and a system of real algebraic curves is pure [impure] if its generic curve is pure [impure]. A topological cremona transformation is pure if its homoloidal net is pure. In the group of all topological cremona transformations of the real plane, the set of pure topological cremona transformations is a proper subgroup. The order of a non-homographic pure topological cremona transformation is of the form $4v+1$, $v>0$, and a classification of all types for $v=1, 2, 3, 4$ is given. The subgroup of all pure topological cremona transformations is generated by those of lowest order, namely 5. No pure topological cremona transformation is the product of cubic (impure) transformations, and hence the group of all topological cremona transformations is not generated by those of lowest order.

G. B. Huff (Athens, Ga.)

6425:

Herszberg, J. A note on a result of Zariski. J. London Math. Soc. 33 (1958), 478–481.

Let algebraic varieties V_i ($i=1, 2, \cdots$) of dimension d ($\geqq 2$) be such that there is a point P_i on each V_i, and (i) V_{i+1} is the derived normal variety of the quadratic transform of V_i with the center P_i and (ii) P_{i+1} dominates P_i (i.e., P_{i+1} is a point on the total transform of the point P_i). Let Ω be the union of the local rings $Q(P_i)$ of the points P_i (over a fixed ground field over which the points are rational). Let s be the least positive integer such that there is an s-dimensional valuation ring of the function field which contains Ω. Let B_s be such a valuation. Then the following three assertions are proved in this article: (I) The center of B_s on V_i is either the point P_i or an s-dimensional subvariety of V_i. (II) If $s\geqq 2$, then for a τ, the center of B_s in every V_i, with $i\geqq\tau$, is s-dimensional. (III) If $s\geqq 2$, then, though B_s may not be unique, the center of B_s on V_i which is s-dimensional is uniquely determined (i.e., independent of B_s).

{Unfortunately, the proof of (III) is not complete.}

M. Nagata (Cambridge, Mass.)

6426:

Zariski, Oscar. On Castelnuovo's criterion of rationality $p_a=P_2=0$ of an algebraic surface. Illinois J. Math. 2 (1958), 303–315.

L'auteur établit un résultat annoncé antérieurement

[Amer. J. Math. **80** (1958), 146–148; MR **20** #3873] au sujet du théorème de Castelnuovo sur la rationalité d'une surface F dont le genre arithmétique p_a et le bigenre P_2 sont nuls. Il restait à établir le théorème dans le cas où, le corps de base $\mathfrak{k}$ étant de caractéristique p, on a $(K^2)=1$, K étant un diviseur canonique de F.

Ce critère de Castelnuovo abstrait est ensuite appliqué à l'étude de la rationalité des involutions planes; on obtient l'énoncé: si $A=\mathfrak{k}(x, y)$ est purement transcendant sur le corps $\mathfrak{k}$ (algébriquement clos) et si Σ est un corps avec $\mathfrak{k}\subseteq\Sigma\subseteq A$, Σ est de degré de transcendance 2 sur $\mathfrak{k}$. On voit en plus que si l'extension $\mathfrak{k}(x, y)/\Sigma$ est séparable, alors Σ est purement transcendant sur $\mathfrak{k}$ et que ce résultat n'est plus vrai si $\mathfrak{k}(x, y)/\Sigma$ n'est pas séparable.

J. Guerindon (Rennes)

6427:

Chow, Wei-Liang; and Igusa, Jun-ichi. Cohomology theory of varieties over rings. Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 1244–1248.

Soient Σ un corps, A un anneau noethérien ayant Σ pour corps des fractions. On appelle modèle affine de A et on note $V(A)$ l'ensemble des anneaux de fractions de A relativement à ses idéaux premiers; un élément de $V(A)$ s'appelle une localité de Σ. Deux modèles affines $V(A_1)$ et $V(A_2)$ sont dits cohérents si $A_1[A_2]$ est un anneau de type fini sur A_1 et sur A_2 et si

$$V(A_1[A_2])=V(A_1)\cap V(A_2).$$

Un modèle de Σ est une réunion finie de modèles affines deux à deux cohérents; alors aucun anneau de valuation de Σ ne domine plusieurs localités d'un même modèle. Un modèle X est muni de la topologie de Zariski: l'ensemble des localités de X contenant un élément donné de Σ est un ouvert. Sur un modèle X on définit à la Serre [Ann. of Math. (2) **61** (1955), 197–278; MR **16**, 953] le faisceau des anneaux locaux, les faisceaux algébriques, les faisceaux cohérents et la cohomologie $H(X, F)$ de X à valeurs dans un faisceau F. Si X est affine et F cohérent on a $H^q(X, F)=0$ pour $q\geqq1$. Si U est un recouvrement affine de X, l'application $H(U, F)\to H(X, F)$ est un isomorphisme. On en déduit l'existence d'une suite exacte de cohomologie lorsqu'on se donne une suite exacte

$$0\to F'\to F\to F''\to 0$$

de faisceaux sur X telle que F' soit cohérent.

Donnons nous maintenant un anneau noethérien de base R, tel que Σ soit extension de type fini du corps des fractions de R. On définit les modèles projectifs à la façon ordinaire. En utilisant le fait que le théorème des syzygies est valable pour les modules gradués sur un anneau de polynômes sur un anneau local régulier, on démontre que, si R est un anneau local régulier, les $H^q(X, F)$ (F faisceau cohérent sur un modèle projectif X) sont des R-modules de type fini.

On suppose enfin que R est un anneau de valuation discrète; soit Ru son idéal maximal; posons $k=R/Ru$. Si X est un modèle projectif sur R, on considère l'ouvert Y de X formé par les localités qui contiennent le corps des fractions K de R, et le modèle projectif Y' sur k obtenu à partir de Y par réduction modulo Ru; étant donné un faisceau cohérent F sur X, notons F_Y sa restriction à Y, et F' le faisceau cohérent $k\otimes_R F$ sur Y'. Pour tout R-module M, on note ${}_uM$ le sous-module des éléments de M annulés par u. On a alors l'égalité

$$\text{(1)}\quad \dim(H^q(Y', F'))-\dim(H^q(Y, F_Y)) =\dim({}_uH^q(X, F))+\dim({}_uH^{q+1}(X, F)).$$

En particulier, en notant χ les caractéristiques d'Euler-Poincaré, on a $\chi(Y', F')=\chi(Y, F_Y)$. Autre application: soient V une variété projective normale, E un fibré vectoriel sur V, (V', E') une spécialisation (d'égales ou inégales caractéristiques) de (V, E) telle que V' soit une variété normale et E' un fibré vectoriel sur V'; si $S(E)$ et $S(E')$ sont les faisceaux des germes de sections de E et E', on a

$$\dim(H^q(V', S(E')))\geqq\dim(H^q(V, S(E))),$$
$$\chi(V', S(E'))=\chi(V, S(E));$$

en particulier V et V' ont même genre arithmétique. Enfin l'aspect pathologique des spécialisations d'inégales caractéristiques [cf. Igusa, Proc. Nat. Acad. Sci. U.S.A., **41** (1955), 964–967; MR **17**, 534] peut être expliqué par la formule (1).

P. Samuel (Clermont-Ferrand)

6428:

Rosenlicht, Maxwell. A note on derivations and differentials on algebraic varieties. Portugal. Math. **16** (1957), 43–55.

For most of this paper, the author sums up known results on derivations, differentials, local derivations and local differentials on algebraic varieties and group varieties. In the final paragraph, letting P be a simple rational point over K of an algebraic variety V defined over k, the author defines a differential $\omega(P)$ of K over k by $D\to\omega_P(D_P)$, where D is a derivation of K over k, and D_P, ω_P are a local derivation and a local differential of V at P (D_P is the one which is induced by D). Then he shows that, when V is a commutative algebraic group variety, $\omega(PQ)=\omega(P)+\omega(Q)$.

T. Matsusaka (Evanston, Ill.)

6429:

Lang, Serge. Divisors and endomorphisms on an abelian variety. Amer. J. Math. **79** (1957), 761–777.

The author gives a direct proof on abelian varieties for Weil's formula $\sigma(\xi\xi')>0$ or, in the notation of this article, $\mathrm{tr}(\alpha\alpha')>0$ [see Weil, *Variétés abéliennes et courbes algébriques*, Actualités Sci. Ind., no. 1064, Hermann, Paris, 1948; *Sur les courbes algébriques et les variétés qui s'en déduisent*, Actualités Sci. Ind., no. 1041, Hermann, Paris, 1948; MR **10**, 621, 262].

The following mapping D_X plays an important role in the present article: Let A and B be abelian varieties and let $H(A, B)$ be the module of homomorphisms of A into B. $N(A)$ denotes the Néron-Severi group of divisors of A (i.e., (group of divisors of A)/(divisors which are algebraically equivalent to zero)). For a divisor X on B, D_X is the mapping from $H(A,B)\times H(A, B)$ into $N(A)$ defined by

$$D_X(\alpha, \beta)\equiv(\alpha+\beta)^{-1}(X)-\alpha^{-1}(X)-\beta^{-1}(X).$$

The important case is that where $A=B$ (in this case, $H(A, B)$ is denoted by $H(A)$) and X is a positive non-degenerate divisor on A.

Properties of the mapping D_X are proved in § 1. In § 2, the trace of $\alpha\in H(A)$ is defined by

$$r\cdot\deg(X^{(r-1)}\cdot D_X(\alpha, \delta))/\deg(X^{(r)}),$$

where $r=\dim A$, $X^{(i)}$ is the numerical equivalence class of the intersection of X with itself i times and δ is the identity in $H(A)$. This definition and the properties of D_X lead to the formula $\mathrm{tr}(\alpha\alpha')>0$ stated above. § 3 concerns a pairing of elements of $H(A)$ into the group of 1-cycles of A modulo numerical equivalence, which is regarded as a dual of D_X, and arrives at the Lefschetz fixed point

formula, stated as follows:

$$\deg(T\cdot\Delta)=d(T)-\mathrm{tr}(\gamma)+d'(T),$$

where T is a correspondence on a curve C and γ is the associated endomorphism of the Jacobian variety. § 4 concerns positivity of symmetric endomorphisms.

M. Nagata (Cambridge, Mass.)

6430:

Severi, Francesco. Fondamenti per la geometria sulle varietà algebriche. Ulteriore contributo alla teoria delle irregolarità. Rend. Accad. Naz. dei XL (4) **8/9** (1957/58), 89–97.

This paper develops in detail the proofs of results, most of which have been announced elsewhere [C. R. Acad. Sci. Paris **244** (1957), 2333–2336; MR **19**, 458], concerning the irregularities of various dimensions of algebraic varieties and their interpretations, analytical, geometrical, and topological. For details of the results we refer to the earlier review. The author makes an interesting conjecture, which seems worth quoting. An irreducible non-singular subvariety W_k (of dimension k) of the non-singular variety V_r (of dimension r) is said to be ordinary if the number of everywhere finite s-ple integrals for $1\leqq s\leqq(k-1)$ is the same for W_k as it is for V_r. The conjecture is that the ordinary subvarieties W_k of V_r are characterised by the relations $R_i(W_k)=R_i(V_r)$, $i=1, \cdots, (k-1)$, where R_i denotes the ith Betti number.

J. A. Todd (Cambridge, England)

LINEAR ALGEBRA

See also 6557, 6558, 6559, 6560, 6711.

6431:

Nilov, G. N. Calculation of a characteristic of a matrix A. Kabardin. Gos. Ped. Inst. Uč. Zap. **12** (1957), 17–20. (Russian)

6432:

Marcus, Marvin; and Moyls, B. N. Linear transformations on algebras of matrices. Canad. J. Math. **11** (1959), 61–66.

Let M_n denote the algebra of square matrices of degree n over the field of complex numbers. The authors are interested in determining necessary and sufficient conditions for a linear transformation of M_n to preserve certain distinguished subsets. Let U_n, H_n, and R_k denote, respectively, the unimodular group, the set of hermitian matrices, and the set of matrices of rank k $(k=1, \cdots, n)$. Let T denote a linear transformation of M_n considered as a vector space of dimension n^2 over the complex numbers. The following results are obtained. 1: $T(R_k)\subseteq R_k$ for all k if and only if there exist invertible matrices U and V such that either $T(A)=UAV$ or $T(A)=UA'V$ for all $A\in M_n$. 2: The following conditions are equivalent: (i) $T(U_n)\subseteq U_n$; (ii) T preserves determinant; (iii) there exist unimodular matrices U and V such that either $T(A)=UAV$ or $T(A)=UA'V$ for all $A\in M_n$. 3: The following conditions are equivalent: (i) T preserves eigenvalues (counting multiplicities) for H_n; (ii) T preserves eigenvalues for M_n; (iii) there exists a unimodular matrix U such that either $T(A)=UAU^{-1}$ or $T(A)=UA'U^{-1}$ for all $A\in M_n$. 4: Suppose $T(H_n)\subseteq H_n$ and that T preserves eigenvalues for H_n. Then the matrix U of (3) must be unitary. 5: If T preserves determinant for H_n, then it does so for M_n.

W. E. Jenner (Lewisburg, Pa.)

6433:

Yen, Chih-ta; et Chen, Jar-sun. Sur les transformations linéaires dans l'espace unitaire admettant une anti-involution. Acta Math. Sinica **8** (1958), 36–52. (Chinese. French summary)

Let L be a finite-dimensional vector space over the complex field K. An anti-involution of L is a mapping $x\to\bar{x}$ such that

$$\overline{x+y}=\bar{x}+\bar{y},\quad \overline{x\lambda}=\bar{x}\bar{\lambda}\ \text{and}\ \bar{\bar{x}}=\iota x,$$

where $x, y\in L$, $\lambda\in K$, and ι is equal to 1 or -1. L will be called a UR-space if it has a nondegenerate hermitian form (x, y) and an anti-involution $x\to\bar{x}$ such that $\overline{(x, y)}=\varepsilon(\bar{x}, \bar{y})$, $\varepsilon=\pm1$. In the usual manner, an endomorphism A of L is defined to be real if it commutes with the anti-involution, and to be symmetric or skew-symmetric if $(xA, y)=(x, yA)$ or $(xA, y)=-(x, yA)$ for all x, y of L. This paper is concerned with the real symmetric and skew symmetric endomorphisms of a UR-space. The authors give the normal forms for such endomorphisms and discuss thoroughly their elementary divisors. As a consequence, it is proved that "Two real symmetric or skew symmetric endomorphisms of a UR-space are equivalent if and only if they are equivalent in the unitary space." Furthermore, the UR-spaces are classified. They fall into four classes depending on the values of ι and ε. The groups of automorphisms of spaces in these four classes coincide, respectively, with the groups of isometries of the symmetric spaces of Cartan types BDI, CI, CII and DIII.

H. C. Wang (Evanston, Ill.)

6434:

Hodges, John H. Scalar polynomial equations for matrices over a finite field. Duke Math. J. **25** (1958), 291–296.

The square matrices under consideration are over the finite field GF(q) of order $q=p^n$. Let $E=E(x)$ be a monic polynomial over GF(q), and let $N(E, m)$ be the number of matrices θ of order m such that $E(\theta)=0$. In this paper the classical theory of the scalar polynomial equation for matrices over a field is combined with a theorem of L. E. Dickson [*Linear groups*, Teubner, Leipzig, 1901; p. 235] concerning commutativity of certain matrices over GF(q), to yield an explicit formula for $N(E, m)$. This formula is expressed in terms of the number $g(t, d)$ of non-singular matrices of order $t\geqq1$ over GF(q^d) given by

$$g(t, d)=q^{dt^2}\prod_{i=1}^{t}(1-q^{-di})=\prod_{i=0}^{t-1}(q^{dt}-q^{di}).$$

The special cases where $E(x)=x^e-1$ and x^3-1 are discussed in detail. The even more special case $E(x)=x^2-1$ has been treated by the author in an earlier paper [Amer. Math. Monthly **65** (1958), 518–520; MR **20** #3163].

A. L. Whiteman (Los Angeles, Calif.)

6435:

Bellman, Richard. Eigenvalues and functional equations. Proc. Amer. Math. Soc. **8** (1957), 68–72.

The determination of the largest eigenvalue of a symmetric matrix of order n is equivalent to the determination of the maximum of the associated quadratic form of n normalized variables. The article gives, for Jacobi matrices and some extensions of them, a sequence of n functions of an increasing number of variables, each of which is maximized in its turn. From the last one the

largest eigenvalue is derived. In an analogous way the smallest eigenvalue can be found. The same principle is applied to an eigenvalue problem of an ordinary differential equation of the second order.

W. H. Muller (The Hague)

6436:

Paasche, Ivan. Zwei Determinantengestalten der Bernoullischen Polynome und ihre Überführung ineinander. Jber. Deutsch. Math. Verein. **61** (1958), Abt. 2, 1–3.

The author represents $S_{k-1}(n)=\sum_{r=1}^{n} r^{k-1}$ in two different forms, each a k by k determinant, one of which is given as Problem 377 on the page following his note. He then finds a linear transformation of the matrix of one determinant into the matrix of the other.

R. D. James (Vancouver, B.C.)

ASSOCIATIVE RINGS AND ALGEBRAS

See also 6675.

6437:

Borevič, Z. I. On the fundamental theorem of the Galois theory for skew-fields. Leningrad. Gos. Ped. Inst. Uč. Zap. **166** (1958), 221–226. (Russian)

The Cartan-Jacobson Galois theory for non-commutative fields is simplified, in fact reduced to a few theorems of non-commutative linear algebra, by the introduction of the notion of rank, replacing that of the order of the group of automorphisms. For two skew-fields K and L let $R=R(L, K)$ be the left-linear space over K of all homomorphisms of the addition group of L into the addition group of K. The rank of a subset A of R is the dimension of the subspace $\{A\}$ generated by A in R. Let P be a sub-skewfield of K and G the group of all automorphisms of K leaving fixed the elements of P. Then K is said to be a Galois extension of P and G the Galois group of K over P if rank $G=(K:P)$, i.e. the dimension of K as right P-modul over P. It is proved: (1) If G is a group of automorphisms of K of finite rank n, and P the sub-skewfield of all G-invariant elements of K, then $(K:P)=n$, K is Galois over P and its Galois group $\bar{G}$ is obtained by completing G, i.e. by adjoining to G all automorphisms of K which are contained in $\{G\}$. (2) If K is Galois over P and $PCLCK$, then K is also Galois over L. Hence the correspondence between the complete subgroups H of the Galois group of K over P and the between-fields L is established. *H. Schwerdtfeger* (Montreal, P. Q.)

6438:

Berman, Gerald; and Silverman, Robert J. Near-rings. Amer. Math. Monthly **66** (1959), 23–34.

In this paper the authors state some justifications for studying near-rings, give a few examples of near-rings which are not rings, indicate some differences between rings and near-rings (e.g., the multiplicative center of a near-ring is not necessarily a near-ring), and obtain several simple results on near-rings by extending standard theorems from ring theory. In particular, the Peirce decomposition and the fundamental relationship between ideals and homomorphisms are generalized to near-rings. The authors also suggest some problems involving near-rings and include a number of references.

W. E. Deskins (East Lansing, Mich.)

6439:

Utumi, Yuzo. A note on an inequality of Levitzki. Proc. Japan Acad. 33 (1957), 249–251.

If R is a ring and M is an R-module, let $j(R)$ designate the nilpotency index of R modulo its radical and $m(M)$ designate the supremum of all numbers r such that M contains a direct sum of r mutually isomorphic submodules. It is proved that for any submodules N and N' of M, $m(N+N')\leq m(N)+m(N')$, provided for every nonzero $x\in M$ there exists a nonzero r-ideal A of R such that $x^{-1}0\cap A=0$. Also, if R is a semisimple I-ring, it is shown that $m(R)=j(R)$. Using these results, the author gives a new proof of a result of Levitzki [Univ. Lisboa. Revista Fac. Ci. A. (2) 3 (1954–1955), 203–207; MR **18**, 7] that for any r-ideals A and B of an I-ring R, $j(A+B)\leq j(A)+j(B)$. *R. E. Johnson* (Northampton, Mass.)

6440:

van Leeuwen, L. C. A. The n-fiers of ring extensions. Nederl. Akad. Wetensch. Proc. Ser. A. **61**=Indag. Math. **20** (1958), 514–521.

Some well-known results on Schreier extensions of rings are proved using the concept of an n-fier [B. Brown and N. H. McCoy, Duke Math. J. **13** (1946), 9–20; MR **7**, 361]. The results have to do mainly with extensions of integral domains [Szendrei, Acta Univ. Szeged. Sect. Sci. Math. **13** (1950), 231–234; MR **12**, 474].

R. E. Johnson (Northampton, Mass.)

6441:

Kertész, A. Correction to my paper "Systems of equations over modules". Acta Sci. Math. Szeged **19** (1958), 251–252.

The formulas given in Theorem 8 [same Acta **18** (1957), 207–234; MR **19**, 1155] for solving a system of linear equations are corrected, as are the corollaries of the theorem. *R. E. Johnson* (Northampton, Mass.)

6442:

Yoshii, Tensho. On algebras of left cyclic representation type. Osaka Math. J. **10** (1958), 231–237.

Let A be an associative finite-dimensional algebra with unit. Let N be the radical of A and $\sum_{i=1}^{n}\sum_{j=1}^{f(i)} Ae_{ij}$ be a direct decomposition of A (as a left A-module) into indecomposable components, with $Ae_{ij}\cong Ae_{i1}=Ae_i$. An algebra A is said to be of left cyclic representation type if every indecomposable left A-module is homomorphic to one of the Ae_i. Theorem: An algebra A is of left cyclic representation type if and only if the following conditions are satisfied: (1) each e_iN has only one composition series; (2) each Ne_j is the direct sum of at most two cyclic left ideals, homomorphic to Ae_ν, each of which has only one composition series. The proof is obtained by means of a series of lemmas in which the possible cases are examined. *D. W. Wall* (Ann Arbor, Mich.)

NON-ASSOCIATIVE RINGS AND ALGEBRAS

6443:

Wesson, James R. On Veblen-Wedderburn systems. Amer. Math. Monthly **64** (1957), 631–635.

The reviewer defined a Veblen-Wedderburn system (V-W system) [Trans. Amer. Math. Soc. **54** (1943), 229–277; MR **5**, 72] as a system closed under addition $x+y$ and multiplication xy satisfying: 1) The elements form an

Abelian group, with identity 0, under addition; 2) for $a\neq 0$, the equation $xa=b$ has a unique solution x; 3) for $a\neq 0$, the equation $ax=b$ has a unique solution; 4) $0a=a0=0$; 5) $(a+b)c=ac+bc$; 6) For $a\neq b$ the equation $xa=xb+c$ has a unique solution x.

It is proved here that axiom 4) is redundant and that 6) is redundant for finite V-W systems. It is shown further that every nonzero element of the additive group has the same order, and hence in the finite case the system is of order p^n, p a prime. Finally it is shown that a V-W system of prime order becomes a field when a unit is introduced. Calculations show that a V-W system of order 8 is a field.

Marshall Hall, Jr. (Columbus, Ohio)

6444:

Wan, Cheh-hsian. On the matrix Lie ring defined by a Hamiltonian or skew-Hamiltonian matrix. Acta Math. Sinica **7** (1957), 451–470. (Chinese. English summary)

Dieudonné [Trans. Amer. Math. Soc. **72** (1952), 367–385; MR **14**, 134] and Hua [Acta Sci. Sinica **2** (1953), 1–58; MR **16**, 6] have discussed the unitary groups over a sfield as well as the subgroups generated by transvections. In this paper, the author studies the corresponding problem for Lie rings. In fact, let K be a sfield with characteristic different from 2 and with an involutory involution $a\to\bar a$. Suppose H to be an invertible $(n\times n)$ Hamiltonian or skew Hamiltonian matrix (i.e., $\bar H'=H$ or $\bar H'=-H$) over K. Two vectors x, y in K^n are called orthogonal if $xH\bar y'=0$, and the dimension of the maximal self-orthogonal linear subspace of K^n is called the index of H. Let $L_n(K, H)$ be the Lie ring of all matrices L such that $LH+H\bar L'=0$, and $TL_n(K, H)$ the subring generated by elements of the form $H\bar v'\lambda v$ where $\lambda=\bar\lambda\in K$ and v is an isotropic vector (i.e., self-orthogonal) in K^n. Assuming that the index of H is positive and that the equation $a+\rho\bar a=0$ has a non-zero solution a in K, where ρ denotes 1 or -1 according as H is Hamiltonian or skew, the author studies the structure of $L_n(K, H)$ and $TL_n(K, H)$. The results are analogous to the group case.

H. C. Wang (Evanston, Ill.)

6445:

Yen, Chih-ta. Sur les sous-algèbres commutatives de dimensions maximales d'une algèbre de Lie semi-simple. Sci. Record (N.S.) **1** (1957), 375–376.

The author announces the following results (it appears that all Lie algebras considered are of finite dimension over the complex numbers). Theorem 2: If L is a semi-simple Lie algebra, then among the commutative subalgebras of L of maximal dimension there exist subalgebras consisting of nilpotent elements only. Theorem 3: For a simple Lie algebra of type $\neq A_1$, A_2, the commutative subalgebras of maximal dimension contain only nilpotent elements. — These theorems complement the results obtained by Mal'cev [Izv. Akad. Nauk SSSR. Ser. Mat. **9** (1945), 291–300; MR **7**, 362]; it is stated that Mal'cev's results are used in the proof of theorem 3, but not of theorem 2.

P. M. Cohn (Manchester)

6446:

Block, Richard. New simple Lie algebras of prime characteristic. Trans. Amer. Math. Soc. **89** (1958), 421–449.

Let G_0, G_1, $\cdots$, G_m be a set of finite-dimensional vector spaces, not all zero, over the prime field Z_p of p elements. Let δ_1, $\cdots$, δ_m be non-zero elements of G_1, $\cdots$, G_m, respectively, and let $\delta=\delta_1+\cdots+\delta_m$ in the direct sum $G=G_0+\cdots+G_m$. The author considers a Lie algebra $L=L(G, \delta, f)$ defined over a field F of characteristic p as follows: As a vector space, L has a basis in 1–1 correspondence $\alpha\leftrightarrow v(\alpha)$ with the elements α of G, $\alpha\neq 0$, $-\delta$. Further, on each G_i there is a non-degenerate bi-additive skew function f_i, with values in F, and in case $i>0$, $f_i(\alpha, \beta)=g_i(\alpha)h_i(\beta)-g_i(\beta)h_i(\alpha)$, where g_i and h_i are additive on G_i with values in F, and where $g_i(\delta_i)=0$. Then one defines

$$v(\alpha)v(\beta)=\sum_{i=0}^{m} f_i(\alpha_i, \beta_i)v(\alpha+\beta-\delta_i),$$

where α_i and β_i are the projections of α and β in G on G_i, and where $\delta_0=0$, $v(0)=0$. Then L is a simple Lie algebra of dimension p^n-2 ($n=$ order of G) unless $G=G_0$, when the dimension is p^n-1. These algebras provide examples of simple Lie algebras over algebraically closed fields of characteristic p with Cartan subalgebras of arbitrarily many distinct dimensions; thus no conjugacy theorem for Cartan subalgebras can be expected. They also carry a non-degenerate symmetric bilinear form t with $t(ab, c)=t(a, bc)$ for all a, b, $c\in L$, and a number of them are restricted. Thus any attempt to extend the classification of the reviewer [Mem. Amer. Math. Soc. no. **19** (1956); MR **17**, 1108] to Lie algebras satisfying these two conditions must expect to encounter serious difficulties. The properties of the class of algebras L with respect to isomorphism to other known simple Lie algebras of prime characteristic are investigated. In each case where an isomorphism with a known algebra is not displayed and where there is equality of dimensions with a known algebra, the algebras are proved nonisomorphic by the fact that their derivation algebras have different structures. This demonstration involves full determination of the derivation algebras. The process whereby the $L(G, \delta, f)$ are defined is a generalization of that utilized by Albert and Frank [Univ. e Politec. Torino Rend. Sem. Mat. **14** (1954–55), 117–139; MR **18**, 52] to define their algebras L_0 and L_δ, and the ones which are restricted can also be viewed as a generalization of the class V_m of the same authors.

G. B. Seligman (Münster)

6447:

Ree, Rimhak. Lie elements and an algebra associated with shuffles. Ann. of Math. (2) **68** (1958), 210–220.

Let R be any associative commutative ring with 1 and $\mathfrak{A}$ the free R-module on the basis elements $a(I)$, where $I=(i_1, \cdots, i_r)$ runs over all r-tuples ($r\geqq 0$) of numbers from 1 to m. The author defines a commutative associative multiplication on $\mathfrak{A}$ by using the notion of a shuffle: If I and $J=(j_1, \cdots, j_s)$ are two sequences of integers from 1 to m, then any arrangement of $i_1, \cdots, i_r, j_1, \cdots, j_s$ in a single sequence $K=(k_1, \cdots, k_{r+s})$ such that the order of the i's and of the j's is preserved is called a shuffle. In this way I and J give rise to $\binom{r+s}{r}$ shuffles, not necessarily distinct. Now the product in $\mathfrak{A}$ is defined by $a(I)a(J)=\sum a(K)$, the sum being taken over all shuffles K of I and J. Next let $\mathfrak{P}$ be the algebra of all power series in the non-commuting indeterminates X_1, $\cdots$, X_m over R. Each element $F=\sum \alpha(I)X_I$ of $\mathfrak{P}$ defines a linear mapping φ: $\mathfrak{A}\to R$ by $\varphi(a(I))=\alpha(I)$. Assume that R admits division by any positive integer. Then the author shows (Theorem 2) that F is a Lie element if and only if $\varphi(a(I)a(J))=0$ for all $I, J\neq\emptyset$. From this result Friedrichs' criterion for Lie elements [cf. K. O. Friedrichs, Comm. Pure Appl. Math. **6** (1953), 1–72; MR **15**, 80; P. M. Cohn, C. R. Acad. Sci. Paris **239** (1954), 743–745; MR **16**, 562; R. C. Lyndon, Michigan Math. J. **3** (1955), 27–29; MR **18**, 659] and the

Dynkin-Specht-Wever formula [E. B. Dynkin, Dokl. Akad. Nauk SSSR **57** (1947), 323–326; W. Specht, Math. Z. **51** (1948), 367–376; F. Wever, Math. Ann. **120** (1949), 563–580; MR **9**, 132; **10**, 425, 676] are eas deduced. — With F and φ as before, $\log F$ may be defined as a power series, provided that $\alpha(\emptyset)=1$. In that case (Theorem 5) $\log F$ is a Lie element if and only if φ is a homomorphism from $\mathfrak{A}$ to R. The resulting relations on α, viz. $\alpha(I)\alpha(J)=\sum\alpha(K)$ (where K runs over all shuffles of I and J) are called the shuffle relations. Theorem 5 is shown to generalize a result of Chen [Ann. of Math. (2) **65** (1957), 163–178; MR **19**, 12], who proved that $\log F$ is a Lie element whenever the coefficients $\alpha(I)$ are given by integration along a path. *P. M. Cohn* (Manchester)

6448:

Dididze, C. E. Non-associative free sums of algebras with an amalgamated subalgebra. Soobšč. Akad. Nauk Gruzin. SSR. **18** (1957), no. 1, 11–17. (Russian)

Let A_α be a collection of algebras (over one and the same field P) and U the intersection of any two distinct algebras so that U is a common subalgebra of all the A_α. Choose a basis Z in U and supplement it for each α by L_α to a basis of A_α. Consider the set M of all non-associative words in $Z\cup(\bigcup_\alpha L_\alpha)$ in which no bracket of length 2 consists of elements of the same A_α. These words are multiplied as in the individual A_α, where applicable, and by juxtaposition otherwise. The algebra A (over P) with the basis M is called the free sum of the A_α with amalgamated U and is denoted as follows: $A=\{\sum_\alpha^* A_\alpha;\, U\}$. It is independent of the bases chosen and becomes the ordinary free sum for $U=0$. An algebra G is called free over the subalgebra U and is denoted by $G=[U;M;\bar{F}]$ if G contains free subalgebras F and $\bar{F}$ such that some set M of free generators of F and some basis Z in U satisfy the following conditions: (i) if $m\in M$ and $u\in U$, then $mu,\ um\in U$; (ii) the non-associative words in $Z\cup M$ in which every bracket of length 2 is a product of elements of M form a basis of the algebra G_0 generated by $U\cup F$; (iii) G is the non-associative free sum of G_0 and $\bar{F}$. G is then independent of the choice of the basis Z in U. The author now proves the following theorem: If A is the non-associative free sum of a subalgebra G with $G=[U;M;\bar{F}]$ and of $\{\sum_\alpha^* A_\alpha;\, U\}$, if U is an ideal in all the A_α, if B is any subalgebra of A, if $V=A\cap U$, and $B_\alpha=A_\alpha\cap B$, then B has a similar representation as the non-associative free sum of a subalgebra G' with $G'=[V;M';\bar{F}']$ and of $\{\sum_\alpha^* B_\alpha;\, V\}$; here M' is a basis of the intersection of B with the subalgebra generated by the A_α over the subspace generated by the B_α. As a corollary of this and further theorems the author shows that if U is a non-zero ideal in each A_α, then the decomposition of A into a non-associative free sum with amalgamated U is unique. The proofs are sketched but a full account is contained in the author's logner paper in Mat. Sb. N.S. **43(85)** (1957), 379–396 [MR **20** #3198]. *K. A. Hirsch* (London)

HOMOLOGICAL ALGEBRA

See also 6414, 6460, 6461.

6449:

Nakayama, Tadasi. On the complete cohomology theory of Frobenius algebras. Osaka Math. J. **9** (1957), 165–187.

Let A be a Frobenius algebra over a field K. In analogy with the case of the group ring of a finite group over the integers, the author defines a complete cohomology theory for A as follows: Let $S(A)=\sum_1^\infty X_{n-1}$ be the usual standard complex for A [H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, N.J., 1956; MR **17**, 1040; Chap. IX]. For $n\geqq 1$ set $X_{-n}{}^0=\mathrm{Hom}_K(X_{n-1},K)$. The algebra A being Frobenius, there is a left module isomorphism between A and $A^0=\mathrm{Hom}_K(A,K)$, which induces an automorphism α of the algebra [T. Nakayama, Ann. of Math. (2) **42** (1941), 1–21; MR **2**, 344]. Then $X_{-n}{}^0$ is turned into a new double A-module X_{-n}, by setting, for any x in $X_{-n}{}^0$ and a in A, $ax=ax$ and $xa=x\alpha(a)$. There results, just as for finite groups, a complete standard complex with augmentation

$$
(1)\qquad \begin{tikzcd}[column sep=small] \cdots \arrow[r] & X_1 \arrow[r] & X_0 \arrow[r] \arrow[d] & X_{-1} \arrow[r] & X_{-2} \arrow[r] & \cdots \\ & & A \arrow[r] & A^0 \arrow[u] & & \end{tikzcd}
$$

The cohomology groups $H^p(A,M)$ of the complex $\mathrm{Hom}_{A^e}(X_p,M)$, $-\infty<p<\infty$, for any two-sided A-module M are introduced and shown to be the usual Hochschild cohomology groups for $p>0$, and the usual Hochschild homology groups for $p<0$, with modified coefficients, however [Cartan and Eilenberg, ibid.], while $H^0(A,M)=M^A/\sigma M=\mathfrak{N}(M)$, where $M^A=\{m$ in M with $am=ma$ for all a in $A\}$ and $\sigma(m)=\sum a_imb_i$, where (a_i) and (b_i) are "dual" bases of A. It is also shown that for $-\infty<p<\infty$, $H^p(A,M)=\mathfrak{N}(C^p(A,M))$, where $C^p(A,M)$ is a certain cochain module made into a double A-module via the differentiations of the complex (1). The author next gives the following generalization of the Hochschild reduction theorem [G. Hochschild, Ann. of Math. (2) **46** (1945), 58–67; MR **6**, 114]: For any integers p, q with $-\infty<p,\ q<\infty$, we have $H^{p+q}(A,M)=H^p(A,C^q(A,M))$. Finally it is noted that all the results carry over to a Frobenius algebra over a commutative ring K which is a free finitely generated K-module, so that this paper provides a genuine generalization of the complete cohomology theory of finite groups.

A. Rosenberg (Evanston, Ill.)

6450:

Nakayama, Tadasi. Note on complete cohomology of a quasi-Frobenius algebra. Nagoya Math. J. **13** (1958), 115–121.

Let A be a quasi-Frobenius algebra over a field. In this note, which is a direct continuation of that reviewed above, the author proves the existence of a complete cohomology theory for A: Let $e_1,\cdots,e_k$, be a complete set of non-isomorphic primitive idempotents of A. Set $e_0=e_1+\cdots+e_k$ and $A_0=e_0Ae_0$, the "core" algebra of A. A_0 is then Frobenius. For any double A-module M, let $M_0=e_0Me_0$; the author constructs a new double A-module $M^\sharp$ from M_0 which in general is not even isomorphic to M as a one-sided A-module. There is a map f of $M^\sharp$ into M. Then it is shown that the groups $H^{-n}(M)=H_{n-1}(A,M^\sharp)$ $(n\geqq 2)$, $H^n(M)=H^n(A,M)$ $(n\geqq 1)$, $H^0(M)=M^A/f(M^\sharp)$, and $H^{-1}(M)=\ker f/$(subspace of $M^\sharp$ generated by $am-ma$) form a complete system of cohomology groups for A in the usual sense [H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, 1956; MR **17**, 1040; p. 235]. They may be obtained from a complete resolution of A. The author then notes that if A is actually Frobenius, $M^\sharp$ as a left A-module is isomorphic to M while as a right A-module it is isomorphic to the module obtained from M by setting $ma=ms(a)$, where s is the Nakayama automorphism of M

[Ann. of Math. (2) **42** (1941), 1–21; MR **2**, 344]. Thus, in this case, exactly the same groups as those found in the earlier paper [#6449] are found. The author finally notes that for any quasi-Frobenius algebra $H^n(M)=H^n(A_0, M_0)$, $-\infty<n<+\infty$, where the right-hand groups are those of the first paper for the Frobenius algebra A_0.

A. Rosenberg (Evanston, Ill.)

6451:

Hochschild, G. Note on relative homological dimension. Nagoya Math. J. 13 (1958), 89–94.

Let R be a ring with identity element 1, and S a subring of R containing 1. In a previous paper [Trans. Amer. Math. Soc. **82** (1956), 246–269; MR **18**, 278] the author defined the relative analogues $\mathrm{Ext}_{(R,S)}^n$ of the functors Ext_R^n and introduced the relative projective dimension $d_{R,S}(M)$ of an R-module M and the relative global dimension $d(R, S)=\sup_M (d_{R,S}(M))$. The absolute dimensions are denoted by $d_R(M)$ and $d(R)$. In this paper the author establishes some relations between the relative and absolute dimensions.

The main general theorem of this type is the following. Let R be right S-flat (i.e., $\mathrm{Tor}_n^S(R, C)=0$ for all left S-modules C and all $n>0$). Then for every R-module M,

$$d_R(M)\leqq d_{R,S}(M)+d(S).$$

If R is also S-projective as a left S-module, then

$$d_S(M)\leqq d_R(M)\leqq d_{R,S}(M)+d_S(M).$$

By using an exterior algebra complex, the author then proves the theorem: Let S be an arbitrary ring with identity, and let $R=S[x_1, \cdots, x_n]$ be the polynomial ring in n variables x_i over S. Then $d(R, S)=n$. Moreover, if M is any non-zero R-module that is annihilated by the x_i's, then $d_{R,S}(M)=n$.

As a consequence, the following result [first proved by Eilenberg, Rosenberg, and Zelinsky, same J. **12** (1957), 71–93; MR **20** #5229] is obtained. For every R-module, $d_S(M)\leqq d_R(M)\leqq n+d_S(M)$ (R and S as above). Moreover, if M is annihilated by the x_i's and is not (0), then $d_R(M)=n+d_S(M)$. Hence $d(R)=n+d(S)$.

D. Buchsbaum (Providence, R.I.)

6452:

Takasu, Satoru. On the change of rings in the homological algebra. J. Math. Soc. Japan 9 (1957), 315–329.

The author is concerned with the following situation. Let Λ and Γ be rings and $\varphi:\Lambda\to\Gamma$ a ring homomorphism. If A is a left Γ-module, then A can also be considered as a left Λ-module. Suppose Y is a Γ-projective resolution of A and X is a Λ-projective resolution of A. Then there exists a Λ-homomorphism $f:X\to Y$ over the identity map of A. If C is a left Γ-module, then f induces a map

$$f^*: \mathrm{Hom}_\Gamma (Y, C)\to\mathrm{Hom}_\Lambda (X, C).$$

The author then defines $\mathrm{Ext}_\varphi^n (A, C)$ to be the homology of the algebraic mapping cylinder of f^* and obtains the usual exact sequence

$$\cdots\to\mathrm{Ext}_\varphi^n (A, C)\to\mathrm{Ext}_\Gamma^n (A, C)$$
$$\to\mathrm{Ext}_\Lambda^n (A, C)\to\mathrm{Ext}_\varphi^{n+1} (A, C)\to\cdots.$$

The first part of the paper is concerned with the formal homological properties of the mapping cylinder and $\mathrm{Ext}_\varphi^n (A, C)$, where the author shows that $\mathrm{Ext}_\varphi^n (A,C)$ could also be defined by taking injective resolutions and is independent of the particular resolutions used. Further, it is shown that if Γ is Λ-projective as a right Λ-module, then

$$\mathrm{Ext}_\varphi^n (A, C)\approx\mathrm{Ext}_\Gamma^{n-1} (K_A, C),$$

where $K_A=\mathrm{Ker}\,(\Gamma\otimes_\Lambda A\to A)$. Similar results are obtained for the tensor product functor.

The paper ends with the following application to the theory of group extensions which was investigated by the reviewer in his thesis (unpublished). Let G be a group, H a subgroup of G, M a G-module and (E, ϕ) an extension of M by H. The author then considers all triples (E^1, ϕ^1, t), where (E^1, ϕ^1) is an extension of M by G and $t:E\to E^1$ such that $t|M$ is the identity and $\phi^1 t=\phi$. The obvious equivalence relation of these relative extensions is then introduced. Defining

$$H^n(G, H:M)=\mathrm{Ext}_\varphi^n (Z, M),$$

where φ is the natural inclusion of the integral group ring $Z(H)$ into $Z(G)$, it is shown that there is a one-to-one correspondence between the equivalence classes of relative extensions as defined above and $H^2(G, H:M)$.

M. Auslander (Waltham, Mass.)

6453:

Kaplansky, Irving. Projective modules. Ann. of Math (2) **68** (1958), 372–377.

The following theorem is proved first of all: if an R-module is a direct sum of countably generated R-modules, so is any direct summand. As a particular case, this yields the main theorem that any projective R-module is a direct sum of countably generated (projective) R-modules. The theorem implies that problems on projective modules may be reduced to the countably generated case. In fact, the author deduces the following typical results: 1) if R is a local ring (i.e., if the non-units of R form an ideal), then any projective R-module P is free; 2) if R is a commutative semihereditary ring, then any projective P is a direct sum of R-modules isomorphic to finitely generated ideals; 3) if R is a regular ring, any projective P is a direct sum of R-modules isomorphic to principal one-sided ideals. It is to be noted that these results are known or easy to see if P is finitely generated, and the difficulties occur in the infinitely generated case.

G. Azumaya (Evanston, Ill.)

GROUPS AND GENERALIZATIONS

See also 6358, 6349, 6423, 6447, 6449, 6450.

6454:

Mihaĭlova, K. A. The occurrence problem for direct products of groups. Dokl. Akad. Nauk SSSR **119** (1958), 1103–1105. (Russian)

J. Nielsen has proved [Math. Scand. **3** (1955), 31–63; MR **17**, 455] that in a free group the following problem can be solved algorithmically: given a word w and a finitely generated subgroup H, to decide whether w is an element of H or not. By way of contrast the author shows that the corresponding problem, which she calls the (strong) occurrence problem and which of course is stronger than the word problem, is insoluble for the direct product of two free groups, each on two generators. The example is constructed by means of Novikov's group with an unsoluble word problem. However, if the direct factors are chosen to be abelian groups with a soluble (strong) occurrence problem, then the direct product also has a soluble (strong) occurrence problem. The same conclusion holds if "strong" is replaced by "weak". Here the weak occurrence problem differs from the strong one in that the required algorithm need not be uniform, but may

depend on the finitely generated subgroup H. Finally: the weak occurrence problem is soluble for the direct product of a group with a soluble weak occurrence problem and a group with a soluble strong occurrence problem and with maximal condition for subgroups.

K. A. Hirsch (London)

6455:

Gol'dina, N. P. Solution of some algorithmic problems for free and free nilpotent groups. Uspehi Mat. Nauk (N.S.) 13 (1958), no. 3(81), 183–189. (Russian)

R. C. Lyndon [Proc. Amer. Math. Soc. 3 (1952), 579–583; MR 14, 242] and A. I. Mal'cev [Mat. Sb. 37(79) (1955), 567–572; MR 17, 345] have shown that the word problem for finitely generated nilpotent groups is soluble. In the present note the author gives an explicit algorithm to decide whether an element of a free group lies in the nth term of the lower central series or not. This gives a solution of the word problem for the free nilpotent groups of class n and of the conjugacy problem for the free metabelian groups.

The algorithm depends on the theory of basic commutators and consists of a finite number of identical transformation by which an element of the free group, expressed in terms of a finite number of free generators, is carried into the (unique) form which exhibits the element (modulo the nth term of the lower central series) as a product of basic commutators of increasing weight not exceeding n. [For further information on this standard form of expressing the elements of a free group see M. Hall, Proc. Amer. Math. Soc. 1 (1950), 575–581; MR 12, 388; and N. P. Gol'dina, Dokl. Akad. Nauk SSSR 111 (1956), 528–530; MR 19, 13.] *K. A. Hirsch* (London)

6456:

Dekker, Th. J. On free products of cyclic rotation groups. Canad. J. Math. 11 (1959), 67–69.

The author proves the following two conjectures of J. de Groot [Canad. J. Math. 8 (1956), 261–262; MR 17, 1107]. Two rotations through equal angles α about intersecting axes are free generators of a free group if $\cos\alpha$ is transcendental. A free product of at most continuously many cyclic groups can be isomorphically represented by a rotation group. [Editor's note: Cf. Balcerzyk and Mycielski, Bull. Acad. Polon. Sci. Cl. III. 5 (1957), 1029–1030; MR 20 #1706.]

H. S. M. Coxeter (Toronto, Ont.)

6457:

Kalužnin, L. A. Central extensions of abelian groups. I. Ukrain. Mat. Ž. 8 (1956), 262–272. (Russian)

Let $\Gamma_1, \cdots, \Gamma_s$ be arbitrary groups. By a $(\Gamma_1, \cdots, \Gamma_s)$-extension (G, G_i, ϕ_i) — in abbreviation, a Γ-extension, we understand the group together with an incomplete normal series

$$G=G_0\supset G_1\supset\cdots\supset G_{s-1}\supset G_s,$$

in which the final group G_s is anti-invariant (i.e., the intersection of all the subgroups conjugate to it is the unit element), and a system of homomorphisms $\phi_1, \cdots, \phi_s$, such that ϕ_i maps G_{i-1} onto Γ_i and has G_i for its kernel. If $G_s=1$, the Γ-extension is called regular. For an arbitrary subgroup H of G we set $H_i=H\cap G_i$, $\bar\phi_i=\phi_i|H_{i-1}$. If $(H, H_i, \bar\phi_i)$ is also a Γ-extension, it is called a sub-extension of (G, G_i, ϕ_i). Two Γ-extensions (G, G_i, ϕ_i) and (G', G_i', ϕ_i') are called isomorphic if there exists an isomorphism $\mu: G\approx G'$ such that $\mu(G_i)=G_i'$ and $\phi_i=\phi_i'\mu|G_{i-1}$. For arbitrary $i=1, \cdots, s$ we consider the set C^i of all mappings of the product $\Gamma^{i-1}=\Gamma_1\times\cdots\times\Gamma_{i-1}$ into the group Γ_i (here $C^1=\Gamma_1$). Each element $u=(u_1, u_2, \cdots, u_s)$ of the product $C^1\times\cdots\times C^s$ induces a substitution on the set Γ^s:

$$u(x_1, \cdots, x_s)=(u_1x_1, u_2(x_1)x_2, \cdots, u_s(x_1, \cdots, x_{s-1})x_s),$$

$x_i\in\Gamma_i$. The totality of the substitutions of the Γ^s obtained in this way constitutes a group A. Those substitutions in A which leave unchanged the first i coordinates of the element $e=(e_1, \cdots, e_s)$, where e_i is the unit element of the group Γ_i, constitute a subgroup A_i. If $u\in A_{i-1}$, then $ue=(e_1, \cdots, e_{i-1}, x, \cdots)$, where the mapping $\psi_i: u\to x$ is a homomorphism of A_{i-1} onto Γ_i with kernel A_i. The subgroup A_s is anti-invariant in A. Consequently, the system (A, A_i, ψ_i) is a Γ-extension. This extension is called the universal extension. It is proved that an arbitrary Γ-extension is isomorphic to some sub-extension of the universal extension (A, A_i, ψ_i). An arbitrary isomorphism between two sub-extensions of the universal extension (A, A_i, ψ_i) is induced by an inner automorphism of A generated by an element of the subgroup A_s. *Z. I. Borevič* (RŽMat 1957 #5359)

6458:

Kalužnin, L. A. Central Γ-subextensions of the complete product of abelian groups. Ukrain. Mat. Ž. 8 (1956), 413–422. (Russian)

A Γ-extension $(\mathfrak{G}, \mathfrak{G}_i, \psi_i)$ [see the article reviewed above] is called central (principal) if the normal series

$$\mathfrak{G}=\mathfrak{G}_0\supset\mathfrak{G}_1\supset\cdots\supset\mathfrak{G}_{s-1}\supset 1$$

is a central (principal) series of the group $\mathfrak{G}$. It is proved that a subextension $(\mathfrak{G}, \mathfrak{G}_i, \psi_i)$ of a complete Γ-extension (U, U_i, ψ_i) is a principal extension if and only if, for arbitrary $i=1, \cdots, s$, all elements of the group $\mathfrak{G}_i$ leave unchanged the first i coordinates of every element of the group Γ. It is also proved that the subextension $(\mathfrak{G}, \mathfrak{G}_i, \psi_i)$ of a complete Γ-extension (U, U_i, ψ_i) is a central extension if and only if the centralizer $\mathfrak{G}'=\mathfrak{Z}(U; \mathfrak{G})$ of the group $\mathfrak{G}$ in the group U is transitive. In this case $(\mathfrak{G}', \mathfrak{G}_i', \psi_i)$, with $\mathfrak{G}_i'=\mathfrak{G}'\cap U_i$, is also a central subextension; $\mathfrak{G}=\mathfrak{Z}(U; \mathfrak{G}')$; and also there exists a unique anti-isomorphism $\nu_\mathfrak{G}$ of $\mathfrak{G}$ onto the group $\mathfrak{G}'$, which is at the same time a Γ-anti-isomorphism of the subextension $(\mathfrak{G}, \mathfrak{G}_i, \psi_i)$ onto the subextension $(\mathfrak{G}', \mathfrak{G}_i', \psi_i)$, in the sense that, for every $i=1, 2, \cdots, s$, $\psi_i(X)=\psi_i(\nu_\mathfrak{G}X)$ for all $X\in\mathfrak{G}_{i-1}$, and which is such that the center of $\mathfrak{G}$ is the kernel of the homomorphism $\sigma_\mathfrak{G}:\mathfrak{G}\to U_s$ defined by the formula $\sigma_\mathfrak{G}(X)=X(\nu_\mathfrak{G}(X))^{-1}$. *B. I. Plotkin* (RŽMat 1957 #7657)

6459:

Fuchs, L. On generalized pure subgroups of abelian groups. Ann. Univ. Sci. Budapest. Eötvös. Sect. Math. 1 (1958), 41–47.

For an abelian group G, a subgroup H and a fixed cardinal $\mathfrak{m}$, the author proves that H has the property "for subgroups F, $G\supseteq F\supseteq H$ and $[F:H]<\mathfrak{m}$ imply the existence of a subgroup K with $F=H\oplus K$", if and only if H has the property "every system of linear equations over H with a set of unknowns of power less than $\mathfrak{m}$ is solvable in H whenever it is solvable in G." Such subgroups H are called $\mathfrak{m}$-pure. The $\aleph_0$-pure subgroups are just the well-known serving or pure subgroups, and, for $\mathfrak{m}>|G|$, the $\mathfrak{m}$-pure subgroups are the direct summands of G. There do exist groups with $\aleph_1$-pure subgroups, but not every union of an ascending chain of direct summands need be $\aleph_1$-pure. An $\mathfrak{m}$-pure subgroup H of a p-group G satisfies $p^\alpha H=p^\alpha G\cap H$ for all ordinals α of power less than $\mathfrak{m}$.

Every subgroup H of G can be extended to an $\mathfrak{m}$-pure subgroup of G of a power not too very high. There exist $\aleph_1$-pure subgroups of free groups which are not direct summands but which are direct sums of infinite cyclic groups. Reference is made to an unpublished paper of of J. Łoś, to St. Balcerzyk [Publ. Math. Debrecen **4** (1956), 357–358; MR **18**, 190], to S. Gacsályi [ibid. **4** (1955), 89–92; MR **16**, 898] and to E. Specker [Portugaliae Math. **9** (1950), 131–140; MR **12**, 587].

F. Haimo (St. Louis, Mo.)

6460:

Baer, Reinhold. Die Torsionsuntergruppe einer Abelschen Gruppe. Math. Ann. **135** (1958), 219–234.

A striking example of the application of homological algebra to abelian group theory is provided by this paper. Indeed, the principal object of investigation is $\mathrm{Ext}(B, A)$, the group of abelian group extensions of A by B. {Another application of this relatively new discipline to abelian group theory is given by Harrison [Ann. of Math. (2) **69** (1959), 366–391; MR **21** #3481].} Endomorphisms,s pecifically monomorphisms on B or epimorphisms on A, are shown to induce epimorphisms of Ext, upon application of properties of exact sequences. The kernels of such induced epimorphisms turn out to be homomorphic images of Ext where A and B are replaced by suitable homomorphic images or subgroups. When these modified Ext are trivial, the induced epimorphisms become automorphisms. Let $[n, G]$ {not the author's notation, but convenient here} be the mapping $x \to nx$ where x ranges over the abelian group G. Then $[n, A]$ and $[n, B]$ each induce $[n, \mathrm{Ext}]$. If $\ker[n, B]=0$ and if $\ker[n, \mathrm{Ext}]=\mathrm{Ext}$, then Ext is trivial. Equivalent to $\mathrm{Ext}=p\,\mathrm{Ext}$ for a prime p is $\ker[p, B]=0$ or $A=pA$. If $\ker[p, B]=0$, then $\ker[p, \mathrm{Ext}(B, X)]=0$ for every abelian group X if and only if for some subgroup U of pB, B/U is free abelian. If $\ker[p, B]=0$ and if A is a torsion group, then the finiteness of B/pB or the countability of B implies that $\ker[p, \mathrm{Ext}]=0$. If B is torsion-free and if A is a torsion group, then the countability of B or the finiteness of each B/pB implies that $\mathrm{Ext}(B, A)$ is torsion-free.

Let Z be the group of infinite sequences of integers and let $Z(p)$ be that subgroup, the components of the elements of which, for each positive integer n, are almost all divisible by p^n. Those p-groups which are homomorphic images of $Z(p)$ are (1) direct sums of groups with division and groups of bounded order. These latter groups are precisely the p-groups T which can be described equivalently by any one of the properties: (2) $\ker[p, B]=0$ implies that $\mathrm{Ext}(B, T)=0$; (3) $\mathrm{Ext}(Z, T)=0$; (4) $\mathrm{Ext}(Z(p), T)=0$; (5) $\ker[p, \mathrm{Ext}(Z, T)]=0$ and (6) $\ker[p, \mathrm{Ext}(Z(p), T)]=0$. Finally, the author shows that a p-group T without (3) has the property that both $\mathrm{Ext}(Z, T)$ and $\mathrm{Ext}(Z(p), T)$ are mixed groups which contain elements of any p-prime power order. Hence $\mathrm{Ext}(Z, T)\neq 0$ if T is a direct sum of cyclic groups which have ascending p-prime power orders. He thus gives a negative answer to an earlier question of his [Ann. of Math. (2) **37** (1936), 766–781; see p. 768 and p. 781, footnote]. Quite independently but simultaneously, J. Erdős has answered the same question with essentially the same example but by using p-adic modules [review below].

F. Haimo (St. Louis, Mo.)

6461:

Erdős, Jenő. On the splitting problem of mixed abelian groups. Publ. Math. Debrecen **5** (1958), 364–377.

For a p-adic module G without elements of infinite height and a basic submodule B, each element of G can be represented, essentially uniquely, as the sum of an infinite series, the terms of which are taken from the direct cyclic summands of B. Define the dimension of G to be the rank of B. Let H be a torsion-free p-adic module of countable dimension. Then any extension of a p-adic torsion module T by H to a p-adic module splits ($\mathrm{Ext}(H, T)=0$) if and only if T is the direct sum of a bounded-order module and a divisible module or H is free. Such splitting takes place for every torsion module T if and only if H is free. Defining a p-adic closure of an abelian group G to be any p-adic module M for which G is a group of generators and for which an independent set of G is independent in M, the author shows that an isomorphism between groups extends to an isomorphism between a pair of their p-adic closures, while an abelian group has a p-adic closure if and only if its torsion subgroup is a p-group. Moreover, an abelian p-group P always splits its abelian extensions by a fixed torsion-free abelian group H if and only if, in the p-adic module situation, P always splits its p-adic module extensions by a p-adic closure of H. Define p-adic dimension of an abelian group with a p-adic closure to be the dimension of this closure. Let H be a torsion-free group of countable p-adic dimension. Then $\mathrm{Ext}(H, P)=0$ for every p-group P if and only if the p-adic closure of H is free. If H is a torsion-free group of countable p-adic dimension for at least one prime p, then $\mathrm{Ext}(H, T)=0$ for every torsion group T if and only if H is free. If H is the group of all sequences of integers, then $\mathrm{Ext}(H, T)\neq 0$, where T is the direct sum of all the distinct cyclic groups of orders which are the powers of p. [Cf. #6460 above; for consistency with the latter, we have written $\mathrm{Ext}(A, B)$ in this review for the author's $\mathrm{Ext}(B, A)$.]

F. Haimo (St. Louis, Mo.)

6462:

Kochendörffer, Rudolf. Über die Fortsetzbarkeit von Faktorensystemen. Math. Nachr. **18** (1958), 173–177.

Suppose that an extension of a finite abelian group A by a group G is given by the homomorphism Γ of G into the automorphism group of A and by the factor system (=f.s.) $c(g', g'')\in A$, $g', g''\in G$. For fixed Γ the class of associated f.s. will be denoted by $\{c(g', g'')\}$ and the finite abelian group of associated G-f.s. under elementwise multiplication by $C(G)$. The unit element of $C(G)$ is the class of splitting f.s. Now let H be a subgroup of G of finite index $i=[G:H]$ and let i be prime to the order of A. Then as Gaschütz has shown [J. Reine Angew. Math. **190** (1952), 93–107; MR **14**, 445] a G-f.s. gives rise, by restriction, to an H-f.s.; here associated G-f.s. yield associated H-f.s. and products of G-f.s. yield products of H-f.s. Hence there is a homomorphism of $C(G)$ into $C(H)$ whose kernel consists of those G-f.s. for which the corresponding H-f.s. splits. It turns out that then $c(g', g'')$ itself must split so that we have an isomorphism of $C(G)$ onto a subgroup $C^*(H)$ of $C(H)$. $C^*(H)$ is, in general, a proper subgroup, and coincides with $C(H)$ if and only if every H-f.s. $c(h', h'')$ can be extended to a G-f.s. This extension, if it exists, is unique. In the present note the author makes two remarks about the posibility of such an extension: (1) If a transversal of H in G can be chosen to be invariant under transformation by the elements of H, and if the homomorphism Γ is the identity, then every H-f.s. can be extended to a G-f.s. (2) If H is normal in G, then an H-f.s. $c(h', h'')$ can be extended to a G-f.s. if and only if it is associated to all the f.s. of the form $\{c^r(r^{-1}h'r, r^{-1}h''r)\}$, where r ranges over a transversal of H in G.

K. A. Hirsch (London)

6463:

Dade, Everett C. Abelian groups of unimodular matrices. Illinois J. Math. 3 (1959), 11–27.

An elementary result by the reviewer states that in the modular group (the integral unimodular matrices of order 2 with A and $-A$ identified) the abelian subgroups are cyclic. The author (this paper is his A.B. thesis) proves the following non-elementary generalization. Let G be an abelian subgroup of the group of integral matrices of order n with determinant 1 and elements taken from an algebraic number field of degree d over the rationals. Then the rank of G is at most $d[n^2/4]$, and the minimum number of generators of the periodic subgroup of G is at most $n-1$. These bounds are attained for each field and each n.

The author proves similar results for the abelian subgroups of the group of unrestricted unimodular matrices, and for orders in algebras over algebraic number fields.

K. Goldberg (Washington, D.C.)

6464:

Gutnik, L. A. On the extension of integral subgroups of some groups. Vestnik Leningrad. Univ. Ser. Mat. Meh. Astr. 12 (1957), no. 19, 47–78. (Russian. English summary)

Let Q_n ($n\geqq 2$) be the real unimodular group. If $n=2p$ is even and M is a non-singular skew-symmetric n-rowed matrix, then the matrices α of Q_n satisfying $\alpha M\alpha'=M$ form a group denoted by $S(M)$; the subgroup consisting of matrices with integer coefficients is denoted by $S^*(M)$. In the special case where $M=\begin{pmatrix} 0 & I \\ -I & 0\end{pmatrix}$, $S(M)$ is the symplectic group and $S^*(M)$ is the modular group. In general M may be taken to be of the form $D=\begin{pmatrix} 0 & D_1 \\ -D_1 & 0\end{pmatrix}$, where D_1 is the diagonal matrix diag $\{d_1, d_2, \cdots, d_p\}$ with $d_1=1$, $d_i|d_{i+1}$ $(1\leqq i\leqq p-1)$. The author proves (i) $Q_n{}^*$ is a maximal discrete subgroup of Q_n [this was proved by Hecke for $n=2$, cf. Petersen, Abh. Math. Sem. Univ. Hamburg 12 (1938), 180–199], (ii) if $S^*(D)$ (where D is in the normal form given above) is a maximal discrete subgroup of $S(D)$, then the non-zero elements of D are square-free, (iii) if the nonzero elements of D are square-free, then there is a unique maximal discrete subgroup $S^{**}(D)$ of $S(D)$ containing $S^*(D)$, and $S^{**}(D)$ consists of all $\alpha\in S(D)$ with $\alpha S^*(D)\alpha=S^*(D)$. In case p is odd, $S^{**}(D)=S^*(D)$, while if p is even, $p=2s$ say, the index $[S^{**}(D):S^*(D)]$ equals 2^k, where k is the number of different prime divisors of d_{s+1}/d_s. In particular, for $D_1=I$ this shows that the Siegel modular group is maximal in the symplectic group. The proofs (involving elementary transformations, partitioned matrices and the common measure of a set of commensurable numbers) are essentially computational, and enable the subgroup $S^{**}(D)$ to be constructed explicitly.

P. M. Cohn (Manchester)

6465:

Dieudonné, Jean. Sur la représentation paramétrique de Cayley. Arch. Math. 9 (1958), 39–41.

The author extends the Cayley parametrization to classical groups over division rings of characteristic 2. (1) Let K be a division ring of arbitrary characteristic, J an involutory anti-automorphism of K, A an $n\times n$ non-singular Hermitian matrix over K of the form $C+{}^tC^J$. Then the equation $U=({}^tC^J-S)^{-1}(C+S)$ gives a one-one correspondence between non-exceptional unitary matrices U (${}^tU^JAU=A$, $I+U$ non-singular) and skew-Hermitian matrices S such that $C+S$ is non-singular. (2) Similarly, if K is commutative, $A=\begin{pmatrix} 0 & I_m \\ -I_m & 0\end{pmatrix}$ and $C=\begin{pmatrix} 0 & I_m \\ 0 & 0\end{pmatrix}$, the formula $U=-({}^tC+S)^{-1}(C+S)$ gives a one-one correspondence between non-exceptional symplectic matrices U and symmetric matrices S such that $C+S$ is non-singular. (3) In (2), let K have characteristic 2. It is well known that every orthogonal matrix for the quadratic form $Q(x)=\sum_1^m x_ix_{m+i}$ is a symplectic matrix for A. It is shown that, conversely, a non-exceptional symplectic matrix $({}^tC+S)^{-1}(C+S)$ is orthogonal if, and only if, S is alternating, i.e., all diagonal elements of S are 0. (1), (2) are proved by specializing a result on rings with an involutory anti-automorphism and (3) is deduced by direct calculation.

G. E. Wall (Sydney)

6466:

Yien, Sze-chien. Linear groups over a commutative integral domain. Sci. Record (N.S.) 1 (1957), 297–300.

This paper announces and outlines the proofs of some results on automorphisms of general rings. Proofs are claimed to be simpler than proofs already reported by C. H. Wan [same Record **1** (1957), no. 1, 5–8; MR **20** #909]. Suppose R is a Dedekind ring (R is commutative, there exists a unique factorization into prime ideals, and also one ideal is an extension of another only if the smaller one is a factor of the larger). Then every automorphism of $GL_n[R]$ has one of the forms

$$A(X)=P^{-1}\chi(X)^\sigma X^\sigma P,\ A(X)=P^{-1}\chi(X)^\sigma(X^\sigma)^{-1}P,$$

where σ is an isomorphism of R, and χ is a homomorphism of $GL_n[R]$ onto the multiplicative semigroup of R such that $\chi(\gamma I)=\gamma^{-1}$ implies $\gamma=1$. Automorphisms of $L_n[R]$ (matrices of determinant 1) are also characterized.

J. L. Brenner (Menlo Park, Calif.)

6467:

Cristescu, Romulus. La notion de composantes dans un groupe dirigé. C. R. Acad. Sci. Paris **247** (1958), 1700–1702.

The author applies the notion of normal manifold and that of projection operators in the book: H. Nakano, *Modulared semi-ordered linear spaces* [Maruzen, Tokyo, 1950; MR 12, 420] to ordered groups with the decomposition property: if

$$x\leqq y+z \text{ and } x, y, z\geqq 0,$$

then we can find $x_1, x_2\geqq 0$ such that $x=x_1+x_2$, $x_1\leqq y$, $x_2\leqq z$. He states a theorem: for a system of normal manifolds Q_λ $(\lambda\in\Lambda)$ of an ordered group $\mathfrak{X}$, if $[Q_\lambda]x\geqq 0$ for all $\lambda\in\Lambda$ implies $x\geqq 0$ for the corresponding projection operators $[Q_\lambda]$ $(\lambda\in\Lambda)$, then $\mathfrak{X}$ is embedded into the ordered group of the product $\prod Q_\lambda$.

H. Nakano (Sapporo)

6468:

Aumann, G. Über die Erweiterung von additiven monotonen Funktionen auf regulär geordneten Halbgruppen. Arch. Math. **8** (1957), 422–427.

A commutative semigroup H has a partial ordering $\lll$ such that $x\lll y$ if and only if $x+z\lll y+z$ for all z in H. A real finite-valued function f is called additive and monotone if $f(x+y)=f(x)+f(y)$ and $x\lll y$ implies that $f(x)\leqq f(y)$, and is called subadditive if $f(x+y)\leqq f(x)+f(y)$. Given a subadditive function p, $\{L; p\}$ denotes the set of functions defined on L such that if $x_i, y_j\in L$ and $z\in H$ then $\sum x_i\lll\sum y_j+z$ implies that $\sum f(x_i)\leqq\sum f(y_j)+p(z)$. A preliminary discussion is given of the extensions of

functions from an arbitrary set of definition L to the smallest semigroup containing L. The main results on extensions are as follows. For any semigroup E, $U(E)$ denotes the set of u such that there is an integer $m \geqq 1$ and a $y \in E$ and $z \in H$ so that $y \ll mu+z$; $V(E)$ is the set of v such that for some integer $m \geqq 1$ and $y_1, y_2 \in E$, $y_1 \ll mv$ and $v \ll y_2$.

It is proved, firstly, that if p is a subadditive function on H, $E \subset Y \subset U(E)$ and f is a real function on Y, then there is an extension of f in $\{U(E); p\}$ if and only if $f \in \{Y; p\}$; and secondly that f on E can be extended to an additive monotonic function on $V(E)$ if and only if it is additive and monotonic on E. The first of these results contains the Hahn-Banach theorem and a generalisation of it due to Bauer [#6646 below] as particular cases.

J. L. B. Cooper (Cardiff)

6469:

Maurer, I. Sur les diviseurs normaux du groupe monomial complet topologisé. Gaz. Mat. Fiz. Ser. A. (N.S.) **10(63)** (1958), 543–546. (Romanian. Russian and French summaries)

For an initial segment M of ordinals $0, 1, 2, \cdots$ and for a fixed group G, let $[G]$ be the group of all functions from M to G. Since each permutation of M induces an automorphism of $[G]$, one can form the relative holomorph $P(G)$ of $[G]$ over the resulting group P of induced automorphisms on $[G]$ and, by using sequences, topologize this holomorph by topologizing $[G]$ and P separately. The author claims that there are precisely three types of normal subgroups of this topologized relative holomorph: (I) relative holomorphs of $[G]$ over certain subgroups of P, the elements of which leave "most" components of $[G]$ fixed; (II) for non-trivial normal subgroups N of G, the relative holomorphs of the corresponding $[N]$'s over the trivial subgroup of P; and (III) the relative holomorphs over the trivial subgroup of P of the constant sequences, with the constant in each case chosen from the center of G.

F. Haimo (St. Louis, Mo.)

6470:

Benado, Mihail. Sur la théorie générale des produits réguliers de Monsieur O. N. Golovine. V. Publ. Sci. Univ. Alger. Sér. A **4** (1957), 111–143.

Earlier works of the author, on the topic of the title, include: Math. Nachr. **14** (1955), 213–234 (1956); **16** (1957), 137–194; C. R. Acad. Sci. Paris, **244** (1957), 1595–1597, 1702–1704; **245** (1957), 267–270 [MR **18**, 871; **20** #1642; **19**, 385, 528]. (This last note is amplified in the paper under review.) For terminology, see the papers mentioned above and their reviews. If an operator group G has two decompositions as a regular product, the author shows that there is a common refinement of a highly symmetrical form if and only if a certain significant subgroup, the first structural center of the two decompositions, is trivial. For some regular product representations, certain normal subgroups are sought such that the corresponding factor groups decompose directly in two ways, once for each decomposition of G, in such a way that these two direct decompositions of the factor group have a common refinement. A pair of regular product decompositions of G leads to a significant normal subgroup of G, admissible under endomorphisms of G associated with the two decompositions, the Fitting center F of G; and G/F decomposes directly in two ways, corresponding to the two representations of G as a regular product, where F turns out to be minimal with respect to these properties.

F. Haimo (St. Louis, Mo.)

6471:

Mackey, George W. Multiplicity free representations of finite groups. Pacific J. Math. **8** (1958), 503–510.

Let α be an involutory anti-automorphism of a finite group G. With a representation L of G, denote by L^α the representation $x \to L(x^\alpha)^*$. Considering intertwining operators and generalizing the arguments of the author's former work [Amer. J. Math. **75** (1953), 387–405; MR **14**, 947], the paper proves the equivalence of the conditions: (a) For each pair L, M of irreducible representations of G (in an algebraically closed field whose characteristic is 0 or an odd prime not dividing the order of G) the Kronecker product $L \otimes M$ is a direct sum of mutually inequivalent irreducible components and L^α and L are equivalent; (b) every double coset in $G_3 = G \times G \times G$ of the diagonal subgroup $\bar{G}_3$ is invariant under α;

(c) $$\sum_{x \in G} \zeta_\alpha(x)^3 = \sum_{x \in G} v(x)^2,$$

where $v(x)$ is the order of the centralizer of x in G and $\zeta_\alpha(x)$ is the number of elements z in G satisfying $z(z^\alpha)^{-1} = x$. This generalizes a result of Wigner [Amer. J. Math. **63** (1941), 57–63; MR **2**, 216] on simply reducible groups (in which α is the operator $x \to x^{-1}$). Further, $\sum \zeta_\alpha(x)^{n+1} \leqq \sum v(x)^n$ for all $n = 1, 2, \cdots$; all become equalities whenever one with $n \geqq 3$ is an equality, which occurs if and only if G is commutative and α is the identity. $\sum \zeta_\alpha(x)^2 = \sum v(x)$ if and only if every class in G is invariant under α.

T. Nakayama (Nagoya)

6472:

Berman, S. D. Representations of groups of order 2^m over an arbitrary field of zero characteristic. Dopovidi Akad. Nauk Ukrain. RSR **1958**, 243–246. (Ukrainian. Russian and English summaries)

L'auteur détermine tous les idempotents minimaux de l'algèbre $L(G, K)$ d'un groupe G d'ordre 2^m sur un corps arbitraire K de caractéristique 0, ainsi que les conditions nécessaires et suffisantes de l'isomorphisme des algèbres $L(G, K)$ et $L(G', K)$ de deux groupes G et G' de cette forme sur un même corps K. La description de ces idempotents et les conditions d'isomorphisme sont très compliquées et se formulent à l'aide de certaines suites principales de G. L'auteur ne donne aucune démonstration.

M. Krasner (Paris)

6473:

Lambek, J. Initial segments of positive semigroups. Trans. Roy. Soc. Canada. Sect. III. (3) **50** (1956), 41–46.

Let M be a non-empty set with a partial binary operation $+$ such that

$$(a+b)+c = a+(b+c)$$

if either side is defined, and $a=b$ if and only if $a \in b+M$ or $b \in M+a$. Then $(M, +)$ is isomorphic with an initial segment of the positive part of a simply ordered group. That group is the additive reals or integers if the following additional postulate holds: if $A \subset M$, then $A+M = \emptyset$ or M or $a+M$ for some $a \in M$.

A. J. Hoffman (New York, N.Y.)

6474:

Maury, Guy. Une caractérisation des demi-groupes noethériens intégralement clos. C. R. Acad. Sci. Paris **247** (1958), 254–255.

The semigroup D is noetherian if it has an identity, is commutative and satisfies the ascending chain condition for ideals. Let D_Z denote the quotient semigroup of quotients of elements of D by elements in the subsemigroup Z of all cancellative elements in D. The element x in D_Z is said to be integral over D if there exists an

element a in Z such that $x^n a \in D$ for all positive integers n. D is integrally closed if any element which is integral over D belongs to D. This paper announces, without proofs, conditions for a noetherian semigroup D to be integrally closed in terms of properties of prime ideals containing elements of Z, and of the integral closure of quotient semigroups determined by these prime ideals.

G. B. Preston (Shrivenham)

6475:

Tamura, Takayuki. Notes on translations of a semigroup. Kōdai Math. Sem. Rep. **10** (1958), 9–26.

This paper continues the earlier investigations of the author [same Rep. **7** (1955), 67–70; MR **18**, 318] into the relations between a semigroup S and its semigroup Φ of right translations (ϕ: $S \to S$ is a right translation of S if $\phi(st)=s\phi(t)$). The first section corrects and extends a theorem of the earlier paper on the right translation semigroup of a zero semigroup. In the next section it is shown that Φ is a group if and only if S is a left group. In the final and main section of the paper the case when Φ is a semilattice is investigated. Let $R=\{\rho_s : s \in S\} \subseteq \Phi$, where $\rho_s(t)=ts$ $(t \in S)$. When R is a semilattice the mapping $s \to \rho_s$ is a homomorphism of S onto R which determines canonically an equivalence E, say, on S such that each E-class is an s-indecomposable subsemigroup of S, i.e., is a subsemigroup admitting no homomorphism onto a non-trivial semilattice. The author also shows, conversely, that for an arbitrary choice of semilattice R there exists a semigroup S such that $S/E \cong R$ and in which the E-classes can each be taken as an arbitrary s-indecomposable semigroup. It is shown that when $S^2=S$, then Φ is a semilattice if and only if R is a semilattice, and that when $S^2 \neq S$, necessary and sufficient conditions for Φ to be a semilattice can be given in terms of the E-classes of S.

G. B. Preston (Shrivenham)

6476:

Wiegandt, Richard. On complete semi-modules. Acta Sci. Math. Szeged **19** (1958), 219–223.

In this paper the author continues his study of complete structures [cf. same Acta **19** (1958), 93–97; MR **20** #2387]. A semi-module, i.e., a commutative regular (=cancellative) semigroup, $(S, +)$, is called complete when it is a direct component of every semi-module which contains S as a normal sub-semi-module. The author asserts that a semi-module S with zero-element is complete if and only if it is divisible, that is, $nS=S$ for every natural number n. This would be analogous to a theorem of R. Baer on abelian groups [Bull. Amer. Math. Soc. **46** (1940), 800–806; MR **2**, 126]. But, as V. R. Hancock has shown in a yet unpublished paper, the statement is incorrect and should read: A semi-module is complete if and only if it is a divisible group. In view of this correction the other results of the paper lose their interest.

R. Artzy (Haifa)

6477:

Iséki, Kiyoshi. On ideals in semiring. Proc. Japan Acad. 34 (1958), 507–509.

An ideal M of a semiring R is called compressed if and only if, for any n, $a_1^2a_2^2\cdots a_n^2 \in M$ implies $a_1a_2\cdots a_n \in M$, and completely prime if $ab \in M$ implies $a \in M$ or $b \in M$. An element $x \in R$ is called a T-element for M if x is expressible as $x_1x_2\cdots x_n$ with $x_1^2x_2^2\cdots x_n^2 \in M$ for some n. The ideal of R generated by all the T-e ements for M is denoted by $T_1(M)$, and, inductively, $T_m(M)=T_1(T_{m-1}(M))$. Then $\bigcup_{m=1}^{\infty} T_m(M)=T^*(M)$ is called the Thierrin radical of M. The author proves that $T^*(M)$ is (i) the intersection of all compressed ideals containing M, and (ii) the intersection of all minimal completely prime ideals of R which contain M. {Note: Although R is assumed to be a semiring, no apparent use is made of the additive operation.}

W. E. Deskins (East Lansing, Mich.)

6478:

Choudhury, A. C. Quasigroups and nonassociative systems. III. Bull. Calcutta Math. Soc. **49** (1957), 9–24.

On considère ici un groupe abélien $C(Q)$, déduit du quasigroupe Q au moyen d'un corps convenablement choisi. L'application T qui projette chaque élément sur son unité à gauche ($T=(a \to b)$ avec $ba=a$) se transfère facilement à ce groupe. Relation entre les puissances de T, les "graphs" [Choudhury, même Bull. **40** (1948), 183–194; **41** (1949), 211–219; MR **10**, 591; **11**, 417] de Q et les endomorphismes de $C(Q)$, application aux ringoïdes. L'A. introduit le concept de "fieldoïde": "A ringoid in which all elements except the relative zeros of weight greater than 1 in the left graphs form a quasigroup, will be called a left fieldoid." Homomorphismes dans les quasigroupes et dans les ringoïdes; treillis des partitions normales. Au no. 6 on donne une démonstration particulièrement simple du théorème déjà rencontré par Trevisan [Rend. Sem. Mat. Univ. Padova **19** (1950), 367–370; MR **12**, 313] et Thurston [Proc. Amer. Math. Soc. **3** (1952), 10–12; MR **13**, 621] que, sur tout quasigroupe, Q, deux relations d'équivalence compatibles avec la loi de composition de Q et cancellables des deux cotés (congruences normales), sont permutables. Une suite est annoncée. Le lecteur corrigera facilement de lui-même quelques fautes d'impression, notamment au dernier paragraphe du no. 6.

A. Sade (Marseille)

TOPOLOGICAL GROUPS AND LIE THEORY

See also 6433, 6445, 6447, 6686, 6704, 6705.

6479:

Mycielski, Jan. Some properties of connected compact groups. Colloq. Math. **5** (1958), 162–166.

This paper contains several results about compact connected groups which are known for Lie groups. The main theorem is that a maximal abelian subgroup of a compact connected group is connected and all maximal abelian subgroups are conjugate.

D. Montgomery (Princeton, N.J.)

6480:

Hartman, S.; and Mycielski, Jan. On the imbedding of topological groups into connected topological groups. Colloq. Math. **5** (1958), 167–169.

A topological group is called a c-group if it is arcwise and locally arcwise connected. This paper proves that every topological group can be imbedded as a closed subgroup of a c-group. Furthermore, it contains some sharper statements on the cardinality of the containing group and on some of its other properties.

D. Montgomery (Princeton, N.J.)

6481:

Ruse, H. S. Tensor extensions of metrisable local Lie groups. J. London Math. Soc. 34 (1959), 5–14.

The author in this paper considers the following general problem. Let L be a Lie algebra over the real field and

let A be a finite dimensional associative commutative algebra over the reals. Then $A \otimes L$ is a Lie algebra and hence corresponds to a Lie group $G(A \otimes L)$. Assume the Lie group $G(L)$ has a left and right invariant non-degenerate Riemann metric; when will a group with Lie algebra $A \otimes L$ have the same property? The author shows that if we restrict ourselves to local groups, the existence of the desired Riemann metric on the local group is implied by the existence of a non-singular bilinear form on A. *L. Auslander* (Bloomington, Ind.)

6482:

Yamaguti, Kiyosi. On algebras of totally geodesic spaces (Lie triple systems). J. Sci. Hiroshima Univ. Ser. A 21 (1957/58), 107–113.

The title of this paper refers to the following situation: In a space with affine connection which is symmetric (Cartan), the tangent space is endowed with a ternary composition ([abc]) satisfying several identities. Any vector space equipped with such an operation is called a Lie triple system. The author shows that the characterizing identities originally given by Jacobson [Amer. J. Math. **71** (1949), 149–170; MR **10**, 426] are redundant and can be reduced to a skew symmetry condition, the Jacobi identity, and the requirement that $x \to [abx]$ be a ternary derivation. The realization of a symmetric space as a totally geodesic subspace of a Lie group corresponds to the realization of $[abc]$ as $[[ab]c]$ by imbedding a Lie triple system in a Lie algebra. The associated Lie triple system characterizes a symmetric space to the same extent that a Lie algebra characterizes its Lie group.

W. G. Lister (Oyster Bay, N.Y.)

6483:

Yamaguti, Kiyosi. On the Lie triple system and its generalization. J. Sci. Hiroshima Univ. Ser. A 21 (1957/58), 155–160.

Suppose the requirements of a symmetric space [see the review above] are relaxed by requiring only a linear connection and the vanishing of the covariant derivative of both the torsion and curvature tensors. The tangent space "infinitesimal algebra" which results is a vector space with a Lie algebra-like binary operation, a Lie triple system-like ternary operation and interrelating identities. In abstract form such a system is called a general Lie triple system. The principal result is the exhibition of a construction imbedding, in a suitable sense, any general Lie triple system in a Lie algebra.

W. G. Lister (Oyster Bay, N.Y.)

6484:

Gottschalk, W. H. Minimal sets: an introduction to topological dynamics. Bull. Amer. Math. Soc. 64 (1958), 336–351.

This invited address contains a survey of recent results in topological dynamics, a subject which is defined as "the study of transformation groups with respect to those properties, wholly or largely topological in nature, whose prototype occurred in classical dynamics." The subject had its beginnings in the work of Poincaré, was taken up vigorously by G. D. Birkhoff, and has received a systematic developement in the recent book by the present author and G. A. Hedlund [Topological dynamics, Amer. Math. Soc. Colloq. Publ., Providence, R.I., 1955; MR **17**, 650]. Some of the topics treated may be suggested by the terms almost periodic, recurrent point, regional recurrence, symbolic trajectory, geodesic flow, regionally mixing. There is a long list of unsolved problems and an extensive bibliography. *P. A. Smith* (New York, N.Y.)

TOPOLOGICAL ALGEBRA

6485:

Kasahara, Shouro. Representation of some topological algebras. I. Proc. Japan Acad. 34 (1958), 355–360.

A topological algebra is an algebra over the complex or real field with a topology compatible with the linear space structure such that the ring multiplication is separately continuous. If X is a locally convex space, then the space $L_s(X, X)$ of all continuous linear transformations on X into itself with the topology of pointwise convergence is an example of a topological algebra. The note under review proves the following. Let E be a Hausdorff topological algebra which satisfies the properties: (i) There exists a non-zero element $a \in E$ such that, for any $u \in E$, there is a scalar λ for which $aua = \lambda a$; (ii) for any non-zero elements u, v of E, there is an element w such that $uwv \neq 0$. Then for a suitable locally convex Hausdorff space X, there is a one-to-one, continuous representation of E onto a subalgebra of $L_s(X, X)$ which includes all linear transformations of finite rank.

The author imposes two additional conditions on E to obtain another representation of a similar nature.

I. Namioka (Ithaca, N.Y.)

6486:

Krule, I. S. Concerning binary relations on connected ordered spaces. Canad. J. Math. 11 (1959), 107–111.

Let X be a connected topological space consisting of more than one point such that the topology is given by an order relation R. The author proves necessary and sufficient conditions for a binary relation on X to be either R or its order dual. *A. Lester* (New Orleans, La.)

FUNCTIONS OF REAL VARIABLES

See also 6562, 6649, 6653.

6487:

Menger, K. Is w a function of u? Colloq. Math. 6 (1958), 41–47.

Let R denote the real line. By a fluent, the author (and Newton) means a function (or mapping?) whose range is in R. If u and w are fluents whose domains are A and B, the author asks for a suitable interpretation for $w = f(u)$, where f is a function (with domain in R). Previously, he has used $w = f(u)$ (rel. π), where π is a mapping from A into B, to mean that the diagram

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "\pi"'] & R \arrow[d, "f"] \\ B \arrow[r, "w"'] & R \end{tikzcd}$$

commutes. He now observes that π does not have to be a function, but may, in fact, be any subset of $A \times B$. Define a mapping ϕ on $A \times B$ into $R \times R$ by $\phi(\alpha, \beta) = (u\alpha, w\beta)$. Then, $w = f(u)$ (rel. π) is to be understood to mean $\phi(\pi) \subset f$. *R. C. Buck* (Stanford, Calif.)

6488:

Mordell, L. J. Integral formulae of arithmetical character. J. London Math. Soc. 33 (1958), 371–375.

Let $f_t(x)$ be definedfor $x \geqq 0$. Assume that $f_t(x)$ satisfies

the functional relation

$$f_i(x)+f_i\left(x+\frac{1}{k}\right)+\cdots+f_i\left(x+\frac{k-1}{k}\right)=f_i^{(k)}f(kx),$$

where $f_i^{(k)}$ is a constant independent of x. Let $a_1, \cdots, a_n$ be a set of relatively prime integers; put $A=a_1a_2\cdots a_n$, $\{x\}=x-[x]$. The author proves that, if the integrals exist, then

$$\int_0^A f_1\left(\left\{\frac{x}{a_1}\right\}\right)f_2\left(\left\{\frac{x}{a_2}\right\}\right)\cdots f_n\left(\left\{\frac{x}{a_n}\right\}\right)dx=$$

$$f_1^{(a_1)}f_2^{(a_2)}\cdots f_n^{(a_n)}\int_0^1 f_1(x)f_2(x)\cdots f_n(x)dx.$$

A special case of this theorem has been proved by Mikolás [Publ. Math. Debrecen **5** (1957), 44–53; MR **19**, 731].

P. Erdős (Birmingham)

6489:

Plotnikov, V. I. Generalized saddle functions. Uspehi Mat. Nauk **13** (1958), no. 5(83), 191–196. (Russian)

A continuous function $z(x, y)$ in a region K is said to be an R-generalized saddle function, $R>0$, if for every region DCK and for every pair of numbers a, b, with $a^2+b^2\geqq R^2$, the function $z(x, y)-ax-by$ assumes its greatest and least values on the boundary of D. The author proves that if $z(x, y)$ is an R-generalized saddle function in the circle $K=[x^2+y^2\leqq 1]$, and satisfies a Lipschitz condition with constant L on the circumference $x^2+y^2=1$, then $z(x, y)$ satisfies a Lipschitz condition with constant L_1 on every circle $K=[x^2+y^2\leqq d^2<1]$, where L_1 depends upon L, R, d. It is stated that this lemma replaces previous ones in the 2-dimensional calculus of variations [H. Lebesgue, Ann. Mat. Pura Appl. **7** (1902), 231–339; T. Radó, Acta Math. Sci. Szeged **2** (1926), 228–253].

L. Cesari (Baltimore, Md.)

6490:

Plamennov, I. Ya. Asymptotic differentiation of functions of two real variables. Mat. Sb. N. S. **45**(87) (1958), 433–454. (Russian)

This is a complete account of results announced [Dokl. Akad. Nauk SSSR **105** (1955), 416–418; MR **17**, 593].

L. C. Young (Madison, Wis.)

6491:

Mikolás, Miklós. Über die höheren Differentialkoeffizienten zusammengesetzter Skalar- bzw. Vektorfunktionen und einige Anwendungen derselben. Ann. Univ. Sci. Budapest. Eötvös. Sect. Math. **1** (1958), 49–65.

When one applies the chain rule to obtain higher derivatives of a function of one or more variables, one soon obtains very complicated expressions of sums. In this article the author has worked out a number of formulas, such as some formulas between higher differential coefficients, formulas for $(\operatorname{tg} x)^{(\nu)}$, for $d^\nu(vx^2+wx+z)^w/dx^\nu$, etc., and also relations to classical orthogonal polynomials, as well as to vectors and tensors.

E. Frank (Chicago, Ill.)

6492:

Mack, Sidney F. Second derivatives on level surface elements. Amer. Math. Monthly **65** (1958), 758–760.

Let $F(x, y, z)$ be constant on a surface element S given by $z=f(x, y)$. Then the second derivatives F_{xx}, F_{xy}, F_{nn} are calculated in terms of curvatures, geodesic torsion, and the Laplacian of F.

W. H. Fleming (Providence, R.I.)

6493:

Zmorovič, V. A. On a formula in the theory of multiple integrals. Dopovidi Akad. Nauk Ukrain. RSR **1958**, 1281–1283. (Ukrainian. Russian and English summaries)

The method of vector analysis is employed to establish the formula for the transformation of a double curvilinear integral, calculated by two simple closed smooth curves C_1 and C_2, into a double surface integral, calculated by two smooth bilateral surfaces S_1 and S_2, stretched respectively on contours C_1 and C_2.

From the author's summary

6494:

Biernacki, M. Sur la convergence des intégrales. Colloq. Math. **6** (1958), 247–249.

Let $a(x)\geqq 0$ and obey $\int_0^\infty \{a(x)\}^{-p}dx<\infty$, where $p>0$. Let $v(x)$ be a solution of $v'+av=1$. Then $\int^\infty [v(x)]^p dx<\infty$. Applying this with $a(x)=y''(x)/y'(x)$, one infers that if $y\in C''[0, \infty]$, y', $y''\geqq 0$ and $\int_0^\infty\{y'/y''\}^p dx$ converges, so does $\int_0^\infty \{y/y'\}^p dx$. The first inequality is related to one discovered by Z. Opial [Bull. Acad. Polon. Sci. Cl. III **5** (1957), 847–853; MR **19**, 1051].

R. C. Buck (Stanford, Calif.)

6495:

Kántor, Sándor. The everywhere continuous, but nowhere differentiable function of Z. Geőcze. Mat. Lapok **8** (1957), 264–267. (Hungarian. Russian and German summaries)

Geőcze's function, defined in 1905 [Az Ungvari Áll. Reálisk. 1904/5. Évi Értesítője] as theli mit of a sequence of piecewise linear functions, served to show that a continuous function need not have a rectifiable graph. As is now shown, it gives also a simple example of a continuous nowhere differentiable function.

F. V. Atkinson (Canberra City)

6496:

Császár, Ákos. Sur la fonction de Z. Geőcze. Mat. Lapok **8** (1957), 268–271. (Hungarian. Russian and French summaries)

The author considers sequences of piece-wise linear functions associated with given sequences M_n, δ_n, such that $0<M_0<M_1<\cdots$, $\delta_n>0$, $\delta_n\to 0$ and $\sum M_n\delta_n<\infty$. A typical function $f_n(x)$ has linear sections of gradients of absolute value $<M_n$, corresponding to x-intervals of lengths $<\delta_n$. The choice $M_n=n$, $\delta_n=n^{-3}$ gives examples similar to Geőcze's function [see the preceding review]. Taking $M_n=q^n$, $\delta_n=2^{-n}$, where $1<q<2$, it is possible to form continuous strictly monotone functions, whose derivative vanishes almost everywhere [cf. F. Riesz and B. Sz.-Nagy, *Functional analysis*, transl. by L. F. Boron, Ungar Publ. Co., New York, 1955; MR **17**, 175; 48–49].

F. V. Atkinson (Canberra City)

6497:

Shukla, U. K. On points of non-symmetrical differentiability of a continuous function. III. Gaṇita **8** (1957), 81–104.

Continuing his studies [Gaṇita **2** (1951), 54–61; **4** (1953), 139–141; MR **15**, 19; **16**, 230] of points of non-symmetric differentiability for a real-valued function on a real interval, the author devotes the present paper to the construction and study of a particular such function f on an interval I. Concerning this function the following is shown: f is continuous and of bounded variation on I; $f'(x)=0$ almost everywhere in I; the points of non-symmetric differentiability of f form a set everywhere dense in I.

T. A. Botts (Washington, D.C.)

6498:
Wang, Gwo-jiunn. Extension of differentiable functions defined on a particular set. Advancement in Math. 4 (1958), 569–573. (Chinese. English summary)

Let P be a perfect set such that whenever a is a point of accumulation of $P\cap(a, +\infty)$ or of $P\cap(-\infty, a)$, it is a point of positive lower density of that set. It is shown that any function f defined and differentiable on P with $|f'(x)|<M$ can be extended to an everywhere differentiable function $\bar{f}$ with $|\bar{f}'(x)|<M$ for all x.

J. C. Oxtoby (Bryn Mawr, Pa.)

6499:
Sard, Arthur. Images of critical sets. Ann. of Math. (2) **68** (1958), 247–259.

Es sei f eine Abbildung einer offenen Teilmenge R von E_m in E_n. Ein Punkt x von R heißt kritisch, wenn f in x differenzierbar und der Rang r_x der Funktionalmatrix von f in x nicht maximal ist. Es bedeute μ_s mit $s\geqq 0$ das s-dimensionale Hausdorffsche Maß und A eine Menge von kritischen Punkten, die die Vereinigung abzählbar vieler Mengen endlichen Maßes μ_s ist. Der Verf. zeigt, daß $\mu_s(f(A))=0$ wird, wenn $r_x<s$ für jedes x aus A. Ist $r_x=0$ überall in A, $s>0$ und $f\in C^q$ mit $q\geqq 1$, so wird $\mu_{s/q}(f(A))=0$. Sodann werden neben anderen die folgenden hinreichenden Bedingungen für $\mu_n(f(A))=0$ aufgestellt, wobei $f\in C^q$ sei. I) Es gibt eine Zahl r mit $r<s$, so daß $r_x=r$ überall in A und $q\geqq(s-r)/(n-r)$. II) $q\geqq s-n+1$. Im Fall II) kann diese Voraussetzung über q nicht abgeschwächt werden, wenn $m\geqq s>n$, wie ein Beispiel zeigt.

K. Krickeberg (Heidelberg)

6500:
Rüssmann, Helmut. Über die Existenz einer Normalform inhaltstreuer elliptischer Transformationen. Math. Ann. **137** (1959), 64–77.

G. D. Birkhoff discussed [Acta Math. **43** (1922), 1–119] real analytic area-preserving mappings

$$x_1=f(x, y)=ax+by+\cdots,$$
$$y_1=g(x, y)=cx+dy+\cdots,$$

with a fixed point at the origin, and showed that, in the elliptic case $|a+d|<2$; $\cos\gamma_0=\frac{1}{2}(a+d)$, one can introduce new variables by formal power series in which the mapping has the normal form

$$x_1=x\cos w-y\sin w,\quad y_1=x\sin w+y\cos w,$$
$$w=\sum_{k=0}^{\infty}\gamma_k(x^2+y^2)^k,$$

provided that γ_0/π is irrational. In the present paper it is proven that this transformation is divergent in general, in the following sense: Representing the mapping with a generating function $H(x_1, y)$, the above mentioned transformation is convergent only if the coefficients H_{kl} of H (considered as independent variables in $|H_{kl}|\leqq 1$) satisfy infinitely many conditions $\Phi_\nu=0$, where the Φ_ν are independent power series in H_{kl} convergent for $|H_{kl}|\leqq 1$. The proof is parallel to Siegel's similar statement on Hamiltonian systems near an equilibrium [Math. Ann. **128** (1954), 144–170; MR **16**, 704].

This result should be compared with Siegel's paper [Ann. of Math. (2) **43** (1942), 607–612; MR **4**, 76] on conformal mappings, where the convergence of a similar transformation into a normal form depends on number-theoretical conditions on γ_0.

J. Moser (Cambridge, Mass.)

6501:
Romanovskiĭ, P. I.; and Vorob'ev, A. V. Boundedness conditions and estimates of growth of semi-additive functions. Moskov. Oblast. Pedagog. Inst. Uč. Zap. **57** (1957), 99–106. (Russian)

Theorems are given about bounds of subadditive functions defined on a subset e of a normed semigroup. For example, if $f(x)>-\infty$ on e and $f(x)\leqq M$ for $x\in e$, $\|x\|\leqq\gamma$; if for some $\frac{1}{2}\leqq\lambda<1$, each $x\in e$ has a representation $x=x_1+x_2$, $\|x_k\|\leqq\lambda\|x\|$, $x_k\in e$; and if $c=\log 2/\log\lambda^{-1}$; then $f(x)\leqq 2M(\|x\|/\gamma)^c$ for $\|x\|\geqq\gamma$.

G. G. Lorentz (Syracuse, N.Y.)

6502:
Vorob'ev, A. V. Semi-additive functions on point-sets of Euclidean n-space. Moskov. Oblast. Pedagog. Inst. Uč. Zap. **57** (1957), 107–120. (Russian)

For functions $f(x)$ subadditive on a subset e of a cone K of the n-dimensional euclidean space, theorems about local boundedness are derived. For example, if $a\in\bar{e}\cap K$ is not too close to the boundary of K, and if the Schnirelman density of e, $\inf_{r>0}[\text{meas}(e\cap K_r)/\text{meas } K_r]$ (K_r is the part of K contained in the sphere of radius r with center in the origin) is sufficiently close to 1, then $f(x)$ is bounded in a neighborhood of a. (For $n=1$, this is given in the paper reviewed above). *G. G. Lorentz* (Syracuse, N.Y.)

6503:
Videnskiĭ, V. S. Application of the theory of integral functions to the construction and investigation of N'-functions complementary to given N'-functions. Dokl. Akad. Nauk SSSR **121** (1958), 202–205. (Russian)

An N' function $M(u)$, $u\geqq 0$ is an integral $M(u)=\int_0^u p(t)dt$, where $p(t)$ is an increasing function with $p(0+)=0$, $p(\infty)=\infty$, $p(t)>0$ for $t>0$. The exact expression of the Young conjugate function $N(u)$ of $M(u)$ may be not easily accessible, but also an equivalent function $N_1(u)$ with $N(\alpha u)\leqq N_1(u)\leqq N(\beta u)$, where α, β are positive constants, defines the same Orlicz space as N. Using elementary theorems about entire functions, the author shows that

$$N_1(u)=\log F(e^u),\quad F(z)=\sum_0^\infty \exp(-M(n))z^n$$

is equivalent to N. He also gives conditions which must be satisfied by $M(u)$ in order that $\limsup_{u\to\infty} e^{-\rho u}N(u)=\sigma$, $\rho>0$, $\sigma>0$, or $\limsup_{u\to\infty} u^{-1}\log N(u)=\rho$, $\rho\geqq 0$.

G. G. Lorentz (Syracuse, N.Y.)

MEASURE AND INTEGRATION

6504:
Martin, N. F. G. Exceptional values of metric density. Proc. Iowa Acad. Sci. **65** (1958), 335–339.

An example of a linear set is given whose density at the point 0 exists and has any given value between 0 and 1.

A. Rosenthal (Lafayette, Ind.)

6505:
Zalgaller, V. A. On a method of introduction of a measure. Vestnik Leningrad. Univ. **13** (1958), no. 7, 49–51. (Russian. English summary)

The author proposes the following general scheme for the construction of a measure in a metric space R. Let S be a system of connected closed sets t in R, and let the empty set O belong to S. Let $\phi(t)$ be defined and non-

negative for every t in S. Denote by $\{P\}$ the system of sets P each of which has at least one representation T_P as a finite union of non-overlapping sets t_i in S, and write $d(T_P)=\max \operatorname{diam} t_i$ ($t_i \in T_P$). Let $\mu_0(P)$ equal lim sup $\sum \phi(t_i)$, for $t_i \in T_P$, as $d(T_P)\to 0$, or, if $d(T_P)$ cannot tend to 0, let $\mu_0(P)=0$. Then, for any set M in R, let

$$\mu(M)=\inf_{G\supset M}\sup_{P\subset G}\mu_0(P),$$

where G is always an open set in R. The author states without proof the following theorem. Suppose that for every $P\in\{P\}$ and every $\varepsilon>0$ there is a $\delta>0$ such that when $d(T_P)<\delta$ it is possible, by further subdivision of every $t_i \in T_P$, to get an arbitrarily fine representation T_P' of P as a finite union of non-overlapping sets $t_j' \in S$ for which

$$\sum_{t_j'\in T_P'}\phi(t_j')\geq \sum_{t_i\in T_P}\phi(t_i)-\varepsilon.$$

Then $\mu(M)$ is a Carathéodory measure. {In all this the term "nonoverlapping" may have either its usual sense (that is, having no common interior points) or a more general sense defined by the author. His definition is rather obscure and, on the face of it, does not agree with his example 2.} The author's scheme includes as special cases various processes used by A. D. Aleksandrov [Dokl. Akad. Nauk SSSR **60** (1948), 1483–1486; MR **10**, 147] in definitions of area and integral curvature for surfaces of bounded curvature. *H. P. Mulholland* (Exeter)

6506:

Hayes, Charles A., Jr. A sufficient condition for the differentiation of certain classes of set functions. Proc. Cambridge Philos. Soc. **54** (1958), 346–353.

This paper is closely connected with a previous memoir of C. A. Hayes, Jr., and C. Y. Pauc [Canad. J. Math. **7** (1955), 221–274; MR **17**, 719]. In this previous memoir a certain "halo" condition was proved to be necessary and sufficient for the differentiation of a fixed measure ψ with respect to another fixed set function φ. Now here a "halo" condition is formulated which guarantees the differentiability of entire classes of set functions. All discussions refer to a metric space S; but (as the author states) they could be formulated in very general settings. Let $\hat{x}$ represent the set whose only member is x. A function F is called a "blanket" if for each x of its domain: (1) $x\in S$ and $F(x)$ is a non-vacuous family of subsets of S; (2) $\operatorname{diam}\beta<\infty$ whenever $\beta\in F(x)$; (3) $\inf_{\beta\in F(x)}\operatorname{diam}(\beta\cup\hat{x})=0$. Moreover, let q be fixed with $1<q<\infty$ and define $\mathfrak{B}^{(q)}$ as the class of those measures ψ which are integrals of non-negative functions f for which $\int_M \{f(x)\}^q d\varphi(x)<\infty$, where φ is a fixed Carathéodory measure and M is a bounded φ-measurable set. Then the author proves his main theorem: F differentiates each member of $\mathfrak{B}^{(q)}$. The proof uses essentially a theorem of the previous memoir. Although the author does not prove the necessity of his condition, he shows by means of an interesting example that in a certain sense his main result is as good as can be obtained. *A. Rosenthal* (Lafayette, Ind.)

6507:

Schell, Hugo. Über einige Probleme der Addition von linearen Punktmengen. J. Reine Angew. Math. **200** (1958), 52–88.

Let $A, B, \cdots$ denote sets of non-negative real numbers;

$$\sum_1^l A_\lambda=\left\{\sum_1^l a_\lambda;\, a_\lambda\in A_\lambda\right\};\quad lA=\left\{\sum_1^l a_\lambda;\, a_\lambda\in A\right\}.$$

Let $\underline{m}(A)$ be the inner Lebesgue measure of A; $\underline{m}_A(\xi)=\underline{m}(A\cap[0,\xi])$. The finite density $\delta(A)$ and the asymptotic density $\delta^*(A)$ are defined through

$$\delta(A)=\inf_{0<\xi<\infty}\underline{m}_A(\xi)/\xi,\ \delta^*(A)=\liminf_{0<\xi\to\infty}\underline{m}_A(\xi)/\xi.$$

{Definition (E. 7) seems to contain a misprint.} The author obtains analogues of theorems of additive number theory.

Mann's and related theorems. Let $C=\sum_1^l A_\lambda$; $\gamma\leq 1$. Theorem 2 improves a result of Raikov [Mat. Sb. N.S. **5** (47) (1939), 425–440; MR **1**, 296]: Suppose

$$(1)\qquad \sum_1^b \underline{m}_{A_\lambda}(\xi)\geq\gamma\xi$$

for all ξ with $0\leq\xi\leq x$. Then $\underline{m}_C(x)\geq\gamma x$ [actually a more general result is proved corresponding to van der Corput's version of Mann's theorem; Nederl. Akad. Wetensch. Proc. **50** (1947), 252–261, 340–350, 429–435; MR **8**, 566; **9**, 79]. Theorem 5 deals with asymptotic densities: If (1) holds for all large ξ, then $\delta^*(C)\geq\gamma$. Both theorems are the best possible in the following sense: Let $0\leq\alpha_\lambda$ [$\lambda=1, 2, \cdots, l$], $\sum_1^l\alpha_\lambda\leq\gamma\leq 1$. Then there are l sets $A_1,\cdots,A_l$ such that $\delta(A_\lambda)=\delta^*(A_\lambda)=\alpha_\lambda$ ($\lambda=1,\cdots,l$) and $\delta(C)=\delta^*(C)=\gamma$ (theorems 4 and 7).

The discussion of Erdös' theorem is based on Kasch's first two papers [Math. Z. **62** (1955), 368–387; **64** (1956), 243–257; MR **17**, 712, 1060]. Let $x>0$ be fixed. Suppose $0\in B$ and hB is everywhere dense in $[0, x]$ for some h. If $\tau\in hB$, let $h(\tau)$ be the smallest k such that $\tau\in kB$. Then $l(\xi)=\lim\inf_{\tau\to\xi}h(\tau)$ is Lebesgue integrable in $[0, x]$. Put $\lambda=\sup_{0<y<x}y^{-1}\int_0^y l(\xi)d\xi$. Let $\alpha=\inf_{0<\xi<x}\underline{m}_A(\xi)/\xi$. After proving analogues of Kasch's "Grundformeln", the author derives from them such results as the following. Theorem 9: Let $C=A+B$. Then

$$\frac{\underline{m}_C(x)}{x}\geq\alpha\left(1+c_1(\alpha)\,\frac{1-\alpha}{\lambda}\right);$$

also

$$\frac{\underline{m}_C(x)}{x}\geq\alpha\left(1+c_1(1-\alpha)\,\frac{1-\alpha}{\lambda}\right)$$

if $\lambda\geq 29/27$. Theorem II: Assume, in addition, that A is measurable. Then

$$\gamma\geq\operatorname{Max}(c_1(\gamma),\, c_1(1-\gamma))\,\frac{\gamma(1-\gamma)}{\lambda}+\alpha.$$

Here

$$c_1(\rho)=(1+\sqrt{\rho}+\rho)/(1+\sqrt{\rho})^2 \text{ and } \gamma=\inf_{0<\xi<x}\underline{m}_C(\xi)/\xi.$$

{The statements of these theorems and the definition on the last two lines of p. 66 do not quite seem to agree with the definitions on p. 67, l. 3ff. Thus the reviewer is not certain whether his interpretation is correct.} Corresponding results are obtained for the asymptotic case.

A final short section deals with differences of sets. *P. Scherk* (Saskatoon, Sask.)

6508:

Bogdanov, Yu. S. Absolute Banach integral and measure. Vestnik Leningrad. Univ. **13** (1958), no. 7, 34–37. (Russian. English summary)

The author considers real-valued functions $x(t)$ that are defined and bounded for $-\infty<t<\infty$ and have period 1. For such functions an axiomatic definition of an "integral" over [0, 1] was given by S. Banach [Théorie des opérations linéaires, Monografie Matematyczne, Tom 1, Warsaw, 1932, Chap. II, § 3, pp. 29–32], but his definition involved arbitrary choices. The author ob-

serves that Banach's definition leads to a unique value for $\int_0 x(t)dt$ if and only if there exists a sequence $\{\alpha_k\}_1^\infty$ of distinct numbers α_k such that the limit as $n\to\infty$ of $n^{-1}\sum_{k=1}^n x(t+\alpha_k)$ exists uniformly in t and is independent of t. (When this is the case he calls $x(t)$ absolutely integrable in Banach's sense.) A corresponding result for measurability of a set is obtained by specializing $x(t)$ to be the characteristic function of a set.

H. P. Mulholland (Exeter)

6509:

Yarošenko, N. S. On a new concept of an integral and its application to Stieltjes' integral. Ukrain. Mat. Ž. 10 (1958), 450–462. (Russian. English summary)

Given a continuous monotone plane curve C and a function $K(x, y)$ on a neighborhood of C, the author defines on C an elementary Stieltjes-type curvilinear integral of a y-differential of $K(x, y)$ and obtains by familiar methods, some simple properties of this integral.

L. C. Young (Madison, Wis.)

6510:

Goblirsch, R. P. An area for simple surfaces. Ann. of Math. (2) 68 (1958), 231–246.

Let B the boundary of a 3-simplex and h a homeomorphism from B into euclidean 3-space E^3. Bing showed [same Ann. (2) 65 (1957), 456–483; MR 19, 300] that there is a sequence $H=\{h_n\}$ of piecewise linear homeomorphisms, $h_n:B\to E^3$, converging uniformly to h. Write $A(h_n)$ for the elementary area of h_n and

$$A(h)=\inf_H \liminf_n A(h_n).$$

Since the Lebesgue area $L(h)$ is defined similarly without requiring h_n to be a homeomorphism, clearly $L(h)\leqq A(h)$. Under the rather stringent hypothesis that the 2-sphere $h(B)$ occupies finite two-dimensional Favard measure, it is shown that $A(h)\leqq 144L(h)$.

W. H. Fleming (Providence, R.I.)

FUNCTIONS OF A COMPLEX VARIABLE

See also 6503, 6533a-c, 6556,6561,6621a-b, 6622.

6511:

Lotockiĭ, A. V. On the asymptotic behaviour of analytic functions. Dokl. Akad. Nauk SSSR 122 (1958), 17–19. (Russian)

The author investigate certain relations between the asymptotic behaviour of

$$(1)\qquad f(z)=\sum_{s=1}^\infty(-1)^{s-1}a_sz^s=a_1z-a_2z^2+a_3z^3-\cdots$$

when $z\to\infty$ along the real positive axis and the analytic properties of $a_s=a(s)$ considered as a function of the complex variable s. If $a(s)$ is rational, $f(z)$ has an analytic continuation along the axis $x\geqq 0$. If $a(s)$ has no poles in $\operatorname{Re} s\geqq 0$, $f(x)$ tends to a finite limit. If $a(s)$ has poles in the right half-plane, $f(x)$ tends to ∞ and the order of growth depends on the abscissa of the last pole to the right. Knowing the poles of $a(s)$, an exact asymptotic formula for $f(x)$ can be established. Similar relations exist not only for rational functions. Some other classes of functions are investigated.

$a(s)$ is said to have the property A (is $\in A$) if it is defined for $s=0, 1, 2, \cdots$, if the series (1) is convergent in some neighborhood of $z=0$, and if $f(z)$ has an analytic continuation along $x\geqq 0$ and tends to a constant c when $x\to\infty$. The author proves: (I) If $a(s)$ is a rational function without poles in $\operatorname{Re} s\geqq 0$, $a(s)$ and $a(s)/\Gamma(s+1)$ are both $\in A$. (II) If $a(s)$ is an entire function of order 1 and type $\sigma<\pi$, $a(s)$ is $\in A$. (III) If $a(s)$ is of order 1 and type $\sigma<\pi/2$, $a(s)$ and $a(s)/\Gamma(s+1)$ are both $\in A$. (IV) If $\varphi(s)$ is $\in A$, this property belongs also to $a(s)=\varphi(s)\mu(s)$, where $\mu(s)$ is a rational function without poles in $\operatorname{Re} s\geqq 0$, or a Dirichlet series $\sum_{k=1}^\infty b_kk^{-s}$ absolutely convergent in $\operatorname{Re} s\geqq 0$, or $\Gamma(\beta)\Gamma(s+\alpha)/\Gamma(s+\beta)\Gamma(\alpha)$ with $\operatorname{Re}\beta>\operatorname{Re}\alpha>0$.

If $a(s)$ is a rational function with poles in $\operatorname{Re} s\geqq 0$ (the abscissas of the poles not being integers, except $s=0$ which can be a pole), let p be the last pole to the right. If the line $\operatorname{Re} s=\operatorname{Re} p$ has no other poles, then (1) has an analytic continuation along $x\geqq 0$ and

$$f(x)\sim\begin{cases} c_{-m}\dfrac{(\log x)^m}{m!}\text{ for } p=0,\\[2ex] c_{-m}\dfrac{\pi}{\sin p\pi}\dfrac{(\log x)^{m-1}x^p}{(m-1)!}\text{ for } \operatorname{Re} p>0,\end{cases}$$

where m is the multiplicity of the pole $s=p$ and c_{-m} the coefficient of $(s-p)^{-m}$ in the meromorphic part of $a(s)$ at this pole.

These asymptotic formulae hold for data similar to those in (IV).

B. A. Amirà (Jerusalem)

6512:

Tumarkin, G. C.; and Havinson, S. Ya. Analytic functions on multiply-connected regions of the class of V. I. Smirnov (class S). Izv. Akad. Nauk SSSR Ser. Mat. 22 (1958), 379–386. (Russian)

A finite simply-connected domain D with a rectifiable Jordan-boundary is called a Smirnov-domain (belongs to the class S) if $\varphi(w)$ gives the conformal mapping of $|w|<1$ on D and $\ln|\varphi'(w)|$ is expressed in $|w|<1$ by the Poisson integral

$$\ln|\varphi'(re^{i\alpha})|=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1+r^2-2r\cos(\theta-\alpha)}\ln|\varphi'(e^{i\theta})|d\theta.$$

The authors generalize the notion of class S to multiply-connected domains. Several ways leading to this generalization are given and their equivalence is established. Properties of analytic functions in multiply-connected domains of class S are investigated.

B. A. Amirà (Jerusalem)

6513:

Uchiyama, Saburô. Systems of n complex numbers with vanishing power sums. J. Fac. Sci. Hokkaido Univ. Ser. I 14 (1958), 29–36.

Let $B(m, n)$ be the number of essentially different systems of n complex numbers $z_1, z_2, \cdots, z_n$ with

$$\sum z_i^\nu=0\ (\nu=m+1, m+2, \cdots, m+n-1).$$

The author evaluates $B(m, n)$ directly in the cases $m=2$, 3; $n=2$, 3. He does not note, however, that these values follow immediately from a general recurrence formula for $B(m, n)$ published by him elsewhere [Proc. Japan Acad. 33 (1957), 10–12; Acta Arith. 4 (1958), 240–246; MR 19, 736; 20 #860].

C. G. Lekkerkerker (Amsterdam)

6514:

Tietz, Horst. Über Teilreihen von Potenzreihen. Math. Ann. 136 (1958), 342.

A method of Bohnenblust [Proc. Nat. Acad. Sci. U.S.A. 16 (1930), 752–754] yields the following result, substantially more general than that which Bohnenblust actually

announced: "An arbitrary partial series of a power series either has a larger convergence radius than the latter or else both series have a singularity in common on the circle of convergence." (A partial series of $\sum a_n z^n$ is $\sum \varepsilon_n a_n z^n$, where $\varepsilon_n=0$ or 1.)

R. M. Redheffer (Los Angeles, Calif.)

6515:

Turán, Paul. Über lakunäre Potenzreihen. Rev. Math. Pures Appl. **1** (1956), no. 3, 27–32.

If $f(z)$ is an integral function, then $M(r)$ denotes the maximum of $|f(z)|$ on the circle $|z|=r$, and $M(r, \alpha, \beta)$ denotes the maximum of $|f(z)|$ on the arc $\alpha \leq \arg z \leq \beta$ of this circle. Under the assumption that the power series for $f(z)$ is a gap series, it has been proved by several authors [cf. F. Sunyer Balaguer, Mem. Real Acad. Ci. Art. Barcelona (3) **29** (1948), 475–516; MR **12**, 88] that $M(r)$ can be estimated above in terms of $M(\rho, \alpha, \beta)$, for $\rho > r$, or at least for $(1+\varepsilon)r > \rho > (1-\varepsilon)r$. In the present paper $M(r)$ is estimated in terms of $M(r, \alpha, \beta)$ itself. Assuming that $f(z)=\sum_1^\infty a_\nu z^{\lambda_\nu}$, $\lambda_\nu \geq \nu^{1+\kappa}$, with a fixed $\kappa>0$, the author shows that

$$\limsup\{\log M(r, \alpha, \alpha+\delta)\}\{\log M(r)\}^{-1}=1$$

if $r\to\infty$, uniformly with respect to α, if δ is fixed $(0<\delta<2\pi)$.

The proof depends on the first main theorem (Satz VII) of the author's book [Eine neue Methode in der Analysis und deren Anwendungen, Akadémiai Kaido, Budapest, 1953; MR **15**, 688].

N. G. de Bruijn (Amsterdam)

6516:

Shenton, L. R. Generalised algebraic continued fractions related to definite integrals. Proc. Edinburgh Math. Soc. (2) **9** (1958), 170–182.

This study is a continuation of the author's work [same Proc. (2) **9** (1953), 44–52; **10** (1954), 78–91; **10** (1956), 134–140; **10** (1957), 153–167, 167–188; MR **15**, 781; **16**, 575; **17**, 844; **19**, 410]. He previously gave an expansion of the form $a_0|V_s, W_{s+1}|\div|U_s, V_{s+1}|$ for the second order continued fraction associated with

$$F(z_1, z_2)=\int_0^\infty \frac{d\psi(x)}{(x+z_1)(x+z_2)},$$

where U_s, V_s, W_s satisfy a fourth-order recurrence relation. Here, simple expressions for U_s, V_s, W_s are given in terms of $\chi_{2s}(z_1)$, $\chi_{2s}(z_2)$, $\omega_{2s}(z_1)$, $\omega_{2s}(z_2)$, where

$$\frac{\chi_{2s}(x)}{\omega_{2s}(x)}=\frac{a_0}{x+c_1}-\frac{a_1}{x+c_2}-\cdots-\frac{a_{v-1}}{x+c_v}.$$

A relation is shown between the recurrence formula for the first order continued fraction and that satisfied by U_s, V_s, W_s. These results are also generalized. It is indicated that this work is closely related to problems discussed in the correspondence between Stieltjes and Hermite.

E. Frank (Chicago, Ill.)

6517:

Myrberg, Lauri. Eine Bemerkung zum Picardschen Satz. Ann. Acad. Sci. Fenn. Ser. A. I, no. 255 (1958), 4 pp.

The author constructs a meromorphic function which proves the following theorem. Let Γ be a closed point-set of capacity zero on the real axis $v=0$ in the complex w-plane $(w=u+iv)$. Then there exist a closed point-set E on the real axis $y=0$ of the z-plane $(z=x+iy)$ and a function $w(z)$ meromorphic outside E, with the following properties: (1) All values of w belonging to Γ are Picard-exceptional values. (2) Every point of E is an essential singularity of $w(z)$. (3) The linear measure of E is zero.

B. A. Amirà (Jerusalem)

6518:

Fil'čakov, P. F. Numerical method of conformal mapping of simple and simply connected regions. Ukrain. Mat. Ž. **10** (1958), 434–449. (Russian. English summary)

The author continues his studies of conformal mapping by means of a sequence of elementary mappings. Numerical examples are given.

A. W. Goodman (Lexington, Ky.)

6519:

Mahovikov, V. I. The mixed problem and conformal mapping. Dopovidi Akad. Nauk Ukrain. RSR **1959**, 125–129. (Ukrainian. Russian and English summaries)

"The author shows that on applying the results of the mixed problem discussed in same Dopovidi **1957**, 431–435 [MR **20** #5853], conformal mapping functions may be constructed approximately."

Author's summary

6520:

Heins, Maurice. On a problem of Heinz Hopf. J. Math. Pures Appl. (9) **37** (1958), 153–160.

Let S be a Riemann surface with a nonabelian fundamental group. The paper deals with the existence problem of nontrivial smooth unbounded covering surfaces of S that are conformal replicas of S. The problem had been orally suggested by H. Hopf in 1952.

The author remarks that surfaces S with finite topological characteristics do not possess such covering surfaces [H. Huber, Thesis, Eidgenössische Technische Hochschule, Zürich, 1954]. For hyperbolic surfaces without finite topological characteristics the problem remains open.

For parabolic surfaces the author establishes the following complete results. Every conformal mapping of S into itself is univalent. The mapping is onto if S has no planar boundary elements. A fortiori, the answer to the Hopf problem is in the negative.

The proofs are based on the Lindelöf principle and on the author's lemma on an asymptotic property of the Green's function [J. Math. Pures Appl. (9) **36** (1957), 305–312; MR **20** #108].

L. Sario (Los Angeles, Calif.)

6521:

Tsuji, Ryōhei. On conformal mapping of a hyperelliptic Riemann surface onto itself. Kōdai Math. Sem. Rep. **10** (1958), 127–136.

This paper is concerned with determining estimates for the maximum of the orders of the groups of the conformal automorphisms of regions of hyperelliptic Riemann surfaces of genus g obtained by the removal of k disjoint disks. Estimates are given for this maximum and the cases where equality is attained are determined. The exact value of the corresponding maximum is determined for finite Riemann surfaces of genus 2. The work makes use of classical results of Hurwitz as well as the recent work of Oikawa [Kōdai Math. Sem. Rep. **8** (1956), 23–30, 115–116; MR **18**, 290, 797] concerning the group of conformal automorphisms of a finite Riemann surface.

M. H. Heins (Urbana, Ill.)

6522:

Šaginyan, A. L. On the theorems of Schottky and Picard. Izv. Akad. Nauk Armyan. SSR Ser. Fiz.-Mat. Nauk **10** (1957), no. 1, 9–19. (Russian. Armenian summary)

Let the function f be holomorphic on the Riemann surface Σ; let Ω denote a domain on Σ, and suppose that Ω contains the origin O. Let L denote a rectifiable curve in

Ω which joins O to the point P with coordinate z. For a point Q on L, let s denote the distance OQ measured along L, and let $\rho(s)$ denote the distance of Q from the boundary $S(\Omega)$ of Ω. Let λ denote the elliptic modular function for the right half-plane, with $\lambda(0)=0$, $\lambda(-i)=1$, $\lambda(\infty)=\infty$, and let ν denote the inverse of λ. Theorem: if a and b are finite exceptional values for f in Ω, then

$$\log\left|\frac{f(z)-a}{b-a}\right|<K\exp 2\int_L \frac{ds}{\rho(s)}+C,$$

where C is a universal constant and

$$K<1+\left|\nu\left(\frac{f(0)-a}{b-a}\right)\right|+\left[4\Re\nu\left(\frac{f(0)-a}{b-a}\right)\right]^{-1}.$$

The theorem yields various analogues of Picard's theorem. Example: For a point P in Ω, let $\delta(P, \Omega)$ denote the supremum of the distances between the set E and the boundary $S(\Omega)$, where E ranges over all domains in Ω that contain both O and P. Suppose that Ω is single-sheeted and lies in $|z|<1$, and that, for some $\varepsilon>0$, the function

$$(\log|f(z)|)\exp[-(\pi+\varepsilon)/\delta^2(z, \Omega)]$$

is unbounded in Ω; then the function f takes all finite values, except possibly one, in Ω.

G. Piranian (Ann Arbor, Mich.)

6523:

Šaginyan, A. L. Sur la construction des fonctions analytiques qui ont certaines singularités. Ann. Acad. Sci. Fenn. Ser. A. I. no. 251/9 (1958), 6 pp.

Let L denote a continuous curve from the origin to ∞, and for each finite point p on L let the arc length $s(p)$ from the origin to p be finite. Corresponding to any positive function $\delta(s)$, let $\Omega(L;\delta)$ denote the union of all disks $|z-p|<\delta(s(p))$ (p on L). If the function f is holomorphic in the domain $\Omega(L;\delta)$, and if for each q ($0<q<1$) there exists at most one Picard exceptional value for f, relative to the subdomain $\Omega(L;q\delta)$, then L is called a Julia path (δ) for f.

Now let $\{L_\alpha\}$ denote a family of locally rectifiable arcs from the origin to ∞, subject to the restriction that for each r the linear measure of the intersection of L_α with the disk $|z|<r$ has an upper bound independent of the parameter α. The author uses a result in his paper reviewed above to construct an entire function for which each path L_α is a Julia path (δ). *G. Piranian* (Ann Arbor, Mich.)

6524:

Prilepko, A. I.; and Suvorov, G. D. An existence theorem for convergent sequences of analytic functions. Uspehi Mat. Nauk 14 (1959), no. 1(85), 215–218. (Russian)

Using results from the generalized theory of prime ends due to Suvorov [Mat. Sb. N.S. 33(75) (1953), 73–100; Amer. Math. Soc. Transl. (2) 1 (1955), 67–93; MR 15, 244; 17, 472], the authors prove the following theorem: Let K be a bounded continuum in the complex w-plane and w_0 a point in the component of the complement of K containing the point at infinity. Then there exists a sequence $\{f_n(z)\}$ of functions analytic and schlicht in $|z|<1$ and with $f_n(0)=w_0$, $f_n'(0)>0$, which converges uniformly in this circle and such that there is a point z_0, $|z_0|=1$, for which the set of all accumulation points of all possible sequences $\{f_n(z_n)\}$, where $|z_n|<1$ and $\lim z_n=z_0$, coincides with K. *J. B. Crabtree* (Hoboken, N.J.)

6525:

Tumarkin, G. C. The behaviour near the boundary of a region of certain sequences of derivatives of analytical functions, converging uniformly within the region. Dokl. Akad. Nauk SSSR (N.S.) 114 (1957), 502–505. (Russian)

Let E be a set of positive measure on the circumference of the unit circle in the complex plane, and let a sequence of functions $f_n(z)$ analytic on the open unit circle be said to satisfy the Hinčin-Ostrovskiĭ conditions [cf. same Dokl. 105 (1955), 1151–1154; MR 17, 1068] if

$$\int_0^{2\pi}\ln^+|f_n(re^{i\theta})|d\theta\leq C,\quad 0<r<1,\quad n=1, 2, \cdots,$$

and the sequence $\{f_n(e^{i\theta})\}$ of angular boundary values converges for all $e^{i\theta}\in E$.

Two theorems are proved. Theorem 1 is: if the sequence $\{f_n(z)\}$ satisfies the Hinčin-Ostrovskiĭ conditions, then from $\{f_n(z)\}$ it is possible to select a subsequence $\{f_{n(k)}(z)\}$ which is uniformly convergent in every region $\Omega_{\theta,\alpha}$ with vertex at any one of almost all points $e^{i\theta}$ of E and with arbitrary angle α, and which has the property

$$\lim_{n(k)\to\infty}\iint_{\Omega_{\theta,\alpha}}|f'(z)-f_{n(k)}'(z)|^2d\omega=0,$$

where $f(z)=\lim_{n\to\infty}f_n(z)$.

The second theorem is: if $\{f_n(z)\}$ converges uniformly in $|z|<1$ to $f(z)$, then a necessary and sufficient condition for selecting a subsequence uniformly convergent on closed subregion $\bar{D}$ of the circle $|z|<1$ bounded by rectifiable Jordan curves containing a subset of a set of measure E given on $|z|=1$, the subset being of measure arbitrarily close to E, is that there should exist a subsequence $\{f_{n(k)}(z)\}$ of $\{f_n(z)\}$ such that, for almost all $e^{i\theta}\in E$,

$$\lim_{n(k)\to\infty}S(f_{n(k)}, \theta)=S(f, \theta)<\infty,$$

where $S_\alpha(f, \theta)=\iint_{\Omega_{\theta,\alpha}}|f'(z)|^2d\omega$.

6526:

Piranian, George. The boundary of a simply connected domain. Bull. Amer. Math. Soc. 64 (1958), 45–55.

This is an account on prime ends of a simply connected domain B which contains fresh definitions and interesting earlier results as well as historical backgrounds. The main purpose of this account is to discuss distribution problems of prime ends. Denote by $U(B)$ the topological space of prime ends of B in the sense of Urysohn [Fund. Math. 6 (1924), 229–235] and by $U_k=U_k(B)$ ($k=1, 2, 3, 4$) the set of prime ends in $U(B)$ which are of the kth kind. Starting from the Carathéodory problem, as to whether the relation $U_2(B)=U(B)$ holds, and related results, the author proposes: Problem 1: to find necessary and sufficient conditions on the decomposition of the unit circle C into four disjoint sets E_k ($k=1, 2, 3, 4$) in order that, under some conformal mapping of the unit disk onto a bounded schlicht domain, the points of each set E_k correspond to prime ends of the kth kind; problem 2: to find necessary and sufficient conditions on the decomposition of C into four disjoint sets E_k ($k=1, 2, 3, 4$) in order that, for some simply connected domain B and some appropriate homeomorphism between C and $U(B)$, each set E_k corresponds to the set $U_k(B)$.

After some preliminary discussions, the author gives a partial solution of the second distribution problem, considering the case where U_4 is empty.

K. Noshiro (Nagoya)

6527:

Shi, Shung-tse. A covering theorem on bounded schlicht functions. Advancement in Math. 2 (1956), 675–677. (Chinese)

6528:

Lyu, Šu-čin. Some results on univalent functions in dissertations written by students in the analysis section of the department of mathematics in the North-West University. Advancement in Math. 3 (1957), 325–334. (Chinese)

6529:

Tang, Sung-shih. The sections of schlicht functions. Advancement in Math. 3 (1957), 468–477. (Chinese. English summary)

6530:

Shah, Tao-shing. Goluzin's number $(3-\sqrt{5})/2$ is the radius of superiority in subordination. Sci. Record (N.S.) 1 (1957), 219–222.

Let $f(z)$, $F(z)$ be regular for $|z|<1$. $f(z)$ is called subordinate to $F(z)$, symbolically $f(z)\prec F(z)$, if there exists a function $\omega(z)$, regular for $|z|<1$, $\omega(0)=0$, $|\omega(z)|<1$, such that $f(z)=F(\omega(z))$. The author proves the following: Suppose $F(z)$ is schlicht, $f(z)\prec F(z)$, $\arg f'(0)=\arg F'(0)$. Then $|f(z)|<|F(z)|$ for $|z|<(3-5^{\frac{1}{2}})/2$, except when $z=0$ or $\omega(z)\equiv z$. This improves an inequality of Goluzin [Mat. Sb. N.S. **29(71)** (1951), 209–224, 593–603; MR **13**, 223, 454; see also his *Geometrische Funktionentheorie*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1957; MR **19**, 735; pp. 328–332]. The number $(3-5^{\frac{1}{2}})/2$ cannot be increased, as shown by Goluzin's example, $F(z)=z/(1-z)^2$, $\omega(z)=z^2$. *E. Reich* (Minneapolis, Minn.)

6531:

Shah, Tao-shing. On the radius of superiority in subordination. Sci. Record (N.S.) 1 (1957), 329–333.

Under the same hypotheses as in the paper above [cf. also the above references] the author shows that $|f'(z)|<|F'(z)|$ for $|z|<3-8^{\frac{1}{2}}$, unless $z=0$ or $\omega(z)\equiv z$. The number $3-8^{\frac{1}{2}}$ cannot be increased either, as shown by an example of Goluzin's. *E. Reich* (Minneapolis, Minn.)

6532:

Alenicyn, Yu. E. On functions without common values and the outer boundary of the domain of values of a function. Mat. Sb. N.S. 46 (88) (1958), 373–388. (Russian)

This article contains the proofs of results announced earlier by the author [Dokl. Akad. Nauk. SSSR **115** (1957), 1055–1058; MR **20** #3997].

A. W. Goodman (Lexington, Ky.)

FUNCTIONS OF SEVERAL COMPLEX VARIABLES, COMPLEX MANIFOLDS

6533a:

Bavrin, I. I. Exact estimates for coefficients. Moskov. Oblast. Pedagog. Inst. Uč. Zap. 57 (1957), 19–24. (Russian)

6533b:

Temlyakov, A. A. An estimate of coefficients. Moskov. Oblast. Pedagog. Inst. Uč. Zap. 57 (1957), 25–27. (Russian)

6533c:

Temlyakov, A. A. An inequality for the coefficients of a double power series. Moskov. Oblast. Pedagog. Inst. Uč. Zap. 57 (1957), 43–44. (Russian)

Let z and w denote two complex variables; let D denote the domain $|z|<1$, $|w|<1$ in zw-space; and suppose that the series $\sum_{m,n=0}^{\infty} a_{mn}z^m w^n$ converges absolutely in D, say to $F(z,w)$. From Cauchy's inequality and from corresponding known results for functions of one variable, the author deduces the following. If $a_{00}=1$, $a_{m0}=0$ for $m\geqq 1$, and $\Re F(z,w)\geqq 0$ in D, then $|a_{mn}|\leqq 2$. If $a_{m0}=0$ for all m, $a_{01}=1$ and $a_{m1}=0$ for $m\geqq 1$, and if for each z in $|z|<1$ the function $F(z,w)$ is schlicht in $|w|<1$, then $|a_{mn}|\leqq en$.

The remainder of paper (a), as well as paper (b), is devoted to conditions under which the inequality

$$|a_{mn}|\leqq 2(m+n)!/m!n!$$

holds. Paper (c) contains a remark concerning a two-variable analogue of Cauchy's inequality.

G. Piranian (Ann Arbor, Mich.)

6534:

Aĭzenberg, L. Extension of Fatou's theorem. Moskov. Oblast. Pedagog. Inst. Uč. Zap. 57 (1957), 11–17. (Russian)

Let v be a convex open set in (complex) (w,z)-space, bounded by a surface $F(w,z,\bar{w},\bar{z})=0$. Here F has continuous second partial derivatives, the first partials vanish simultaneously almost nowhere on the boundary, and there is a constant k_0 for which almost no point of the boundary has any plane $w=k_0z+b$ (b arbitrary) as a supporting plane. It is shown that, if f is analytic and bounded in v, f has a non-tangential finite limit at almost every boundary point. *J. L. Doob* (Urbana, Ill.)

6535:

Look, K. H. Unitary geometry in the theory of functions of several complex variables. Advancement in Math. 2 (1956), 567–662. (Chinese)

6536:

Wermer, John. The hull of a curve in C^n. Ann. of Math. (2) 68 (1958), 550–561.

If S is a compact subset of C^n, the space of n complex variables, its hull $h(S)$ is defined to be the set of all x in C^n with the following property: to every polynomial P in n variables there corresponds a point y in S such that $|P(x)|\leqq|P(y)|$. Then $h(S)$ is compact (it is the maximal ideal space of the algebra which is the uniform closure of the polynomials on S) and contains S. The author studies the nature of $h(S)$ when S is an arc or a simple closed curve, with the aid of his recent work on function algebras [same Ann. **67** (1958), 45–71, 497–516; MR **20** #109, #3299].

If $y_1, \cdots, y_n$ are analytic functions in a disc D: $|\lambda|<r$, and if the map

$$\lambda\to Y(\lambda)=(y_1(\lambda), \cdots, y_n(\lambda))$$

is a homeomorphism of D, then the set $Y(D)$ is said to be an element through the point $Y(0)$. A set Σ in C^n is an analytic surface if each point p on Σ, except perhaps for points in a discrete subset Σ_0 (called multiple points), has a neighborhood U in C^n such that $U\cap\Sigma$ is an element through p, while each p in Σ_0 has a neighborhood U in C^n such that $U\cap\Sigma$ is the union of a finite set of elements through p.

Suppose now that $\varphi_1, \cdots, \varphi_n$ are analytic functions on the unit circle T which together separate points on T,

and suppose that the derivative of φ_1 has no zero on T. Let Γ be the simple closed curve consisting of the points

$$(\varphi_1(u), \cdots, \varphi_n(u)) \quad (|u|=1).$$

Theorem: Under these conditions, either $h(\Gamma)=\Gamma$, or $h(\Gamma)-\Gamma$ is an analytic surface with at most finitely many multiple points; the second alternative occurs if and only if

$$\int_{|u|=1} P(\varphi_1(u), \cdots, \varphi_n(u))\varphi_1'(u)du=0$$

for every polynomial P in n variables.

Next, suppose that $\psi_1, \cdots, \psi_n$ are analytic functions on the unit interval I which together separate points on I, and suppose that the derivative of ψ_1 has no zero on I. Let L be the arc consisting of the points

$$(\psi_1(t), \cdots, \psi_n(t)) \quad (0 \leqq t \leqq 1).$$

Theorem: Under these conditions, $h(L)=L$; in fact, every continuous function on L is the uniform limit of polynomials on L.

These theorems are complemented by the following examples: (a) It was previously known that there are arcs in C^n ($n \geqq 2$) whose hulls are at least 2-dimensional; (b) the author now shows that there is a simple closed curve in C^6 whose hull is at least 4-dimensional (it contains the cartesian product of two spheres). Thus, the analyticity of the parametrizations cannot be dropped from the hypotheses of the above theorems, although the possibility is open that it can be replaced by a weaker assumption.

W. Rudin (New Haven, Conn.)

6537:

Baily, Walter L., Jr. The decomposition theorem for *V*-manifolds. Amer. J. Math. **78** (1956), 862–888.

Pour un ouvert $\tilde{U}$ d'un espace topologique séparé, un triple $\tilde{U}=\{U, G, \varphi\}$ est appelé un ensemble uniformisant local (en abrégé e.u.l.) si: U est un voisinage ouvert connexe de l'origine dans R^n; G un groupe fini de transformations C^∞ de U; φ une application continue de U sur $\tilde{U}$ telle que $\varphi\circ\sigma=\varphi$ pour tout $\sigma \in G$, l'application induite $U/G \to \tilde{U}$ étant un homéomorphisme. Une V-variété C^∞ est un espace topologique séparé $\mathscr{V}$ muni d'une famille $\mathscr{F}$ de e.u.l. sur les éléments $\tilde{U}$ d'un système fondamental d'ouverts satisfaisant à des conditions de compatibilité. Définition analogue pour les V-variétés analytiques complexes [Satake, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 359–363; MR **18**, 144]. Un V-fibré sur $\mathscr{V}$, de groupe Γ et de fibre F, est défini, sur une famille $\mathscr{F}$ suffisamment fine, de la façon suivante: pour chaque e.u.l. $\tilde{U} \in \mathscr{F}$, on donne le fibré $B_U=U \times F$ et un anti-isomorphisme h_U de G dans un groupe d'applications fibrées de B_U sur lui-même, tel que, si b est un point de la fibre de $x \in U$, alors $h_U(g)b$ soit dans la fibre de $g^{-1}x$ ($g \in G$), avec des conditions de compatibilité relatives aux changements de cartes. On peut construire une métrique pour un V-fibré B à fibre vectorielle, d'où, en prenant pour B le V-fibré tangent, une métrique riemannienne sur $\mathscr{V}$ et les opérateurs $*$, δ et Δ pour les formes différentielles sur $\mathscr{V}$ avec leurs analogues pour les formes à coefficients dans un V-fibré à fibre vectorielle complexe de dimension 1, sur une V-variété complexe. Soit $\mathscr{V}$ une V-variété C^∞ compacte, de dimension réelle n, munie d'une métrique riemannienne C^∞. Soit M le module des sections C^∞ d'un V-fibré C^∞ B à fibre vectorielle R^N muni d'une métrique C^∞ $\mathfrak{a}$. Sur un ouvert U appartenant à un e.u.l., $h_U(g)^{-1}$ est une application C^∞ $\mathscr{M}_g$ de U dans la variété des matrices carrées, d'ordre N, non singulières. Soit ∇ un endomorphisme de M défini par un opérateur différentiel fortement elliptique sur tout U qui commute avec $\mathscr{M}_g$, pour tout $g \in G$. Un produit scalaire global est défini sur les sections de B à l'aide de la métrique $\mathfrak{a}$ et on désigne par ∇^* l'adjoint de ∇. Soit $\mathscr{H}$ le sous-espace des fonctions Φ telles que $\nabla\Phi=0$. Théorème de décomposition: Si ∇ est auto-adjoint, on a: (1) $M=\nabla M \oplus H$. De plus, $\mathscr{H}$ est de dimension finie. Dans le cas où B est le V-fibré des formes différentielles de degré k, le théorème de de Rham-Satake [note citée], joint au résultat précédent, entraîne: l'espace vectoriel de cohomologie réelle (de Čech) de dimension k est isomorphe à l'espace vectoriel des formes harmoniques de degré k (théorème de Hodge généralisé). De même, sur une V-variété complexe, l'espace des formes harmoniques de type (r, s) à coefficients dans un fibré est isomorphe à l'espace de $\bar{\partial}$-cohomologie des formes différentielles de type (r, s) (théorème de Kodaira généralisé). La technique de la démonstration de (1) est une généralisation de celle de Kodaira [Ann. of Math. (2) **50** (1949), 587–665; MR **11**, 108], basée elle-même sur des méthodes de Hadamard; elle repose sur le théorème local de régularité suivant: Sur un voisinage U d'un point ξ de R^n, si la fonction Φ de norme finie est orthogonale à toutes les fonctions $\nabla^*\Psi$ où Ψ est C^∞, à support compact et de norme finie sur U, alors, la fonction Φ est C^∞, de norme finie, et $\nabla\Phi=0$ sur un certain voisinage de ξ contenu dans U.

P. Dolbeault (Bordeaux)

6538:

Baily, W. L. On the imbedding of *V*-manifolds in projective space. Amer. J. Math. **79** (1957), 403–430.

Soit $\mathscr{V}$ une V-variété analytique complexe compacte [#6537 ci-dessus]. S'il existe, sur $\mathscr{V}$, un espace fibré F à fibre vectorielle complexe de dimension 1 dont la classe caractéristique contient une forme différentielle fermée, définie positive, de type (1, 1), alors il existe une application holomorphe birégulière Φ de $\mathscr{V}$ dans un espace projectif complexe $\mathfrak{S}$ tel que Φ soit un homéomorphisme de $\mathscr{V}$ sur une variété algébrique de $\mathfrak{S}$; l'application Φ induit un isomorphisme de l'anneau local des fonctions holomorphes en $x \in \mathscr{V}$ sur l'anneau local de $\Phi(\mathscr{V})$ en $\Phi(x)$. Démonstration: La V-variété $\mathscr{V}$ possède une métrique kählérienne du fait de l'existence de F; la démonstration de l'algébricité d'une variété de Hodge [K. Kodaira, Ann. of Math. (2) **60** (1954), 28–48; MR **16**, 952] est ensuite généralisée, mais, $\mathscr{V}$ n'étant pas une véritable variété, un théorème local d'immersion doit être établi. Applications: Les espaces suivants sont des variétés algébriques: (1) quotient supposé compact d'un domaine borné de C^n par un groupe discontinu d'automorphismes analytiques; (2) quotient d'une variété algébrique projective sans singularité par un groupe fini de transformations analytiques complexes; (3) variété décrite par Satake [Proc. internat. symposium on algebraic number theory, Tokyo-Nikko, 1955, pp. 107–129, Science Council of Japan, Tokyo, 1956; MR **18**, 731] obtenue en compactifiant le domaine fondamental du groupe modulaire de Siegel de dimension 2 qui agit dans l'espace des matrices symétriques complexes d'ordre 2 dont la partie imaginaire est définie positive (résultat annoncé sans démonstration).

P. Dolbeault (Bordeaux)

SPECIAL FUNCTIONS

See also 6776.

6539:

Jarden, Dov. New formulae for Fibonacci numbers and associates. Riveon Lematematika **12** (1958), 33–35. (Hebrew)

The formulas are

$$U_k = \sum_{i=0}^{2k-1} (-1)^{(i-1)(i-2)/2} \binom{k+[(i-1)/2]}{i} U_{[3i/2]},$$

$$V_k = -\sum_{i=0}^{2k-1} (-1)^{(i-1)(i-2)/2} \binom{k+[(i-1)/2]}{i} V_{[3i/2]},$$

where U_k, V_k are the Fibonacci and associated Fibonacci numbers. *E. G. Straus* (Los Angeles, Calif.)

6540:

∗Hancock, Harris. Lectures on the theory of elliptic functions: Analysis. Dover Publications, Inc., New York, 1958. xxiv+498 pp. $2.55.

An unaltered republication of the first edition [Wiley, New York, 1910].

6541:

Fempl, Stanimir. Über die Amplituden der elliptischen Normalintegralen III Gattung für welche sich solche Integrale auf elliptische Normalintegrale I und II Gattung reducieren. Univ. Beogradu. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. no. **18** (1958), 7 pp. (Serbo-Croatian. German summary)

La note représente la continuation de la recherche de l'auteur [Srpska Akad. Nauka. Zb. Rad. **50** Mat. Inst. 5 (1956), 61–116; MR **19**, 739] relative aux amplitudes des intégrales elliptiques du type indiqué par le titre. *M. Tomić* (Belgrade)

6542:

Koschmieder, Lothar. Elliptische Funktionen als Turánsche Folgen. Jber. Deutsch. Math. Verein. **60** (1957), Abt. 2, 3–6.

Let Φ denote the sequence of functions $\varphi_m(u)$ $(m=0, 1, 2, \cdots)$ of the real variable u. If the Hankel (recurrent) determinants

$$\begin{vmatrix} \varphi_{n-1}(u) & \varphi_n(u) \\ \varphi_n(u) & \varphi_{n+1}(u) \end{vmatrix} \quad (n=1, 2, 3, \cdots)$$

are all negative for a non-empty set S of values of u which is independent of n, then Φ is called a Turán sequence on S. The author shows that certain sequences of Jacobian elliptic functions are Turán sequences, while others are not. On the assumption that the modulus k satisfies the condition $0<k<1$, it is proved that if $\alpha\neq 0$ and β are any two real numbers, the two sequences (I) $\mathrm{sn}(\alpha mu+\beta)$ and (II) $\mathrm{cn}(\alpha mu+\beta)$ are each Turán sequences on the set of all real values of u, except at the points $u_r=2rK/\alpha$, with integral r, where the determinants vanish. Special values of α, β, and k are considered. It turns out, in particular, that adopting the familiar Glaisher notation, the sequences (III) $\mathrm{cd}mu$ and (IV) $\mathrm{sd}mu$ are also Turán sequences on the set of all real values of $u\neq u_r$, while the sequences (V) $\mathrm{dn}mu$ and (VI) $\mathrm{nd}mu$ are not. *W. Seidel* (Notre Dame, Ind.)

6543:

Koschmieder, Lothar. Turánsche Determinanten, mit elliptischen Funktionen gebildet. Math. Nachr. **18** (1958), 265–273.

The investigation in the preceding paper is extended to the remaining six Jacobian functions and it is concluded that none of the sequences (VII) $\mathrm{ns}mu$, (VIII) $\mathrm{nc}mu$, (IX) $\mathrm{dc}mu$, (X) $\mathrm{ds}mu$, (XI) $\mathrm{sc}mu$, (XII)$\mathrm{cs}mu$ is a Turán sequence. *W. Seidel* (Notre Dame, Ind.)

6544:

Fempl, S. Some properties of Heuman's function Λ_0. J. Math. Phys. **37** (1958), 137–142.

The function under consideration is a normal elliptic integral of the third kind. Heuman completely investigated this function in the case that the number of independent variables is reduced to two. This special function was expressed by elliptic integrals of the first and second kind. An addition formula was also derived by Heuman. Here, a multiplication formula for this function is derived, involving elliptic integrals of the first kind and Fourier series. From this multiplication formula some special values of this function are derived. *M. J. O. Strutt* (Zürich)

6545:

Kumar, Ram. Certain infinite series expansions connected with generalised Hankel-transform. Gaṇita **8** (1957), 1–7.

The transform in question is

$$\psi_{\lambda,\mu,\nu}(x) = \int_0^\infty (xy)^\nu J_\lambda{}^\mu(xy) g(y)\, dy,$$

$$J_\lambda{}^\mu(x) = \sum_{r=0}^\infty \frac{(-x)^r}{r!\,\Gamma(1+\lambda+\mu r)}, \quad \mu>0.$$

The author derives various identities for functions ψ with different indices, such as

$$\psi_{\lambda,\mu,\nu}(x) = \mu^{-1}\Gamma(\lambda/\mu) \sum_{r=0}^\infty \frac{1}{\Gamma(\lambda/\mu+r+1)} \psi_{\lambda+\mu r-1,\, \mu,\, \nu+r}(x).$$

He then deduces a number of identities that do not involve the ψ's explicitly, such as

$$\sum_{r=0}^\infty \frac{1}{r!} \left(x^{1-\lambda/\mu} \frac{d}{dx}\right)^m \left[x^{(\lambda/\mu)+r_\beta-x}\right] = \mu^{-m} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-m)} x^{(\lambda-m)/\mu},$$

$$\sum_{r=0}^\infty \frac{(1-s)^r}{r!} x^{\frac{1}{2}r} K_{-2\lambda-r+n-1}(2x^{\frac{1}{2}}) = s^{\frac{1}{2}n-\lambda-\frac{1}{2}} K_{-2\lambda+n-1}(2(xs)^{\frac{1}{2}}).$$

R. P. Boas, Jr. (Evanston, Ill.)

6546:

Ford, F. A. J. On certain indefinite integrals involving Bessel functions. J. Math. Phys. **37** (1958), 157–161.

The author starts from four indefinite integrals, the integrands of which are products of modified Bessel functions of the first kind, of orders unity and zero, and of exponential functions. These integrals occur in the solution of some problems of mathematical physics. He proves relationships between the integrals and then discusses them for a number of special cases, giving a list of 12 references. *M. J. O. Strutt* (Zurich)

6547:

Parodi, Maurice. Équations intégrales et équations du type de Mathieu. J. Math. Pures Appl. (9) **37** (1958), 45–54.

Starting from Mathieu's differential equation the author constructs an integral equation of Volterra type and then proves that every solution of the latter equation is a solution of Mathieu's differential equation. The solution of the Volterra type integral equation is straightforward by iteration. Next the author considers a linear homogeneous partial differential equation of the second

order, akin to Mathieu's type, and proceeds to construct an integrodifferential equation, the solution of which affords solutions of the differential equation under consideration. The integrodifferential equation is of Volterra type and may be solved by an iteration process. The proof of the equivalence of its solutions to those of the differential equation is given. Actual series expressions are derived for the solution. *M. J. O. Strutt* (Zurich)

6548:

Meligy, A. S. On Whittaker functions. J. London Math. Soc. 33 (1958), 456–457.

The author gives a contour integral representation for

$$(1)\qquad I_{k,m}(z)=\frac{1}{\Gamma(2m+1)}M_{k,m}(z),$$

where $M_{k,m}(z)$ is Whittaker's confluent hypergeometric function. He introduces an "extra" parameter α into the integrand of this representation by decomposing one of its factors as

$$(2)\qquad (1+z/t)^k=(1+z/t)^\alpha(1+z/t)^{k-\alpha}.$$

Developing the last term on the right-hand side of (2) in powers of $(z/t)(1+z/t)^{-1}$ and integrating term by term then establishes the identity

$$(3)\qquad I_{k,m}(z)=\sum_{r=0}^{\infty}(-)^r\frac{(k-\alpha)\Gamma(k-\alpha+\frac{1}{2}r)z^{\frac{1}{2}r}}{r!\Gamma(k-\alpha-\frac{1}{2}r+1)}\times I_{\alpha,m+\frac{1}{2}r}(z).$$

T. Erber (Chicago, Ill.)

6549:

Jackson, Margaret. A note on the reducibility of the bilateral hypergeometric series $_3H_3$. J. London Math. Soc. 33 (1958), 475–476.

Whipple [Proc. London Math. Soc. (2) **30** (1930), 81–94] has studied the reducibility of $_3F_2$ to series of the type $_2F_1$ with argument -1. The author uses her formula [J. London Math. Soc. **24** (1949), 238–340; MR **11**, 246] connecting $_3H_3$ and two series $_3F_2$ to express a special $_3H_3$ in terms of two $_2F_1$ with argument -1. By taking special values of the parameters, six of Whipple's results follow from the formula established here.

A. Erdélyi (Pasadena, Calif.)

6550:

Hapaev, M. M. Expansion of hypergeometric and degenerate hypergeometric functions in series of Bessel functions. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **1958**, no. 5, 17–22. (Russian)

The author obtains expansions of hypergeometric and confluent hypergeometric functions in terms of modified Bessel functions. He uses these expansions to obtain asymptotic formulae for the functions so expanded, valid for large values of certain of the parameters appearing in these functions. *P. G. Rooney* (Toronto, Ont.)

6551:

Bhonsle, B. R. On some results involving Legendre polynomials. Gaṇita 8 (1957), 9–16.

From Rainville's series for Legendre polynomials $P_n(x)$ [E. D. Rainville, Bull. Amer. Math. Soc. **51** (1945), 268–271; MR **6**, 211], specialized to the form

$$2^{\frac{1}{2}n}(1+x)^{\frac{1}{2}n}P_n(2^{-\frac{1}{2}}(1+x)^{\frac{1}{2}})=\sum_{k=0}^{n}\binom{n}{k}P_k(x),$$

various formulas are obtained involving $P_n(2^{-\frac{1}{2}}(1+x)^{\frac{1}{2}})$. Many of these are immediately deduced from known integral formulas involving Legendre polynomials [A. Erdélyi, Tables of integral transforms, vol. II, pp. 276–278, McGraw-Hill, New York-Toronto-London, 1954; MR **16**, 468]. The author's previous paper [Proc. Amer. Math. Soc. **8** (1957), 10–14; MR **18**, 730] is in close connection with the present paper.

C. A. Swanson (Vancouver, B.C.)

ORDINARY DIFFERENTIAL EQUATIONS

See also 6435, 6638, 6655.

6552:

Molčanov, N. N. Investigation of solutions of the differential equation $dy/dx=Y(x,y)/X(x,y)$, where $Y(x,y)$, $X(x,y)$ are rational polynomials in the real variables x, y in the neighborhood of a singular point. Uspehi Mat. Nauk 13 (1958), no. 6(84), 105–110. (Russian)

The author seeks a solution of the given equation given implicitly by $u(x,y)=0$, where u is an integral of

$$(1)\qquad Xdy+Ydx.$$

The integration is done by means of an integrating factor, M, which is a solution of

$$(2)\qquad X\frac{\partial M}{\partial x}+Y\frac{\partial M}{\partial y}=\left[\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right]M.$$

It is shown that there exists a neighborhood of a singular point of the given equation for which either (1) has an integral of the form $\Psi_1(x,y)+c\Psi_2(x,y)$, with Ψ_1 and Ψ_2 holomorphic in this neighborhood or (2) has a solution of the form $M=\exp\{\theta_1(x,y)/\theta_2(x,y)\}$, with θ_1 and θ_2 holomorphic in this neighborhood. The method used involves the introduction of a third variable, z, and finding a function $V(x,y,z)$ for which θ_1 and θ_2 are solutions of $V(x,y,z)=0$. Equation (2) gives an equation for V which satisfies Weierstrass theorem that there exists a neighborhood in which V is the product of a polynomial in z (with holomorphic coefficients) by a non-vanishing function. The possible cases for the discriminant of various polynomials are then investigated. Finally, $u(x,y)=0$ is shown to be a boundary curve for certain regions of holomorphism. *S. Hoffman* (Hartford, Conn.)

6553:

★Golubew, W. W. Vorlesungen über Differentialgleichungen im Komplexen. Hochschulbücher für Mathematik, Bd. 43. VEB Deutscher Verlag der Wissenschaften, Berlin, 1958. xii+312 pp. DM 32.00.

A translation by Christa and Lothar Berg from the Russian [Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950; MR 13, 131]. Chapters 7 and 8, on the theory of automorphic functions, have been omitted in this translation.

6554:

Bandić, Ivan. On a recurrent linear differential equation of the second order. Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske Ser. II. 12 (1957), 181–187. (Serbo-Croatian summary)

The theorem proved in this paper can be formulated as follows: Let X be any function of x and let y be any solution of the differential equation $y''-yX=0$. If p is any function of x and if y_1 is defined by

$$y_1=\frac{py'-yp'}{(p^2X-p''p)^{\frac{1}{2}}},$$

then y_1 satisfies a differential equation $y_1''-X_1y_1=0$,

where X_1 is a function explicitly given in terms of X, p, and their first and second derivatives.

H. Levi (New York, N.Y.)

6555:

Šestakov, A. A. Existence theorems for integral and critical straight lines of a homogeneous system of n differential equations ($n\geq 3$). Uspehi Mat. Nauk **14** (1959), no. 1(85), 245–248. (Russian)

Consider the system

(1) $$\dot{x}_s = X_s^{(m)}(x_1, \cdots, x_n),$$

where $s=1, \cdots, n$; $m\geq 3$; $X_s^{(m)}$, $s=1, \cdots, n$, is a form of integral degree m, where m does not depend upon s. The real solutions of the system of algebraic equations

(2) $$x_1 : X_1^{(m)} = \cdots = x_n : X_n^{(m)}$$

are straight lines which are the integral straight lines of (1). If a solution of (1) tends to the origin or to infinity asymptotic to a straight line L, then the author calls L a critical line of (1).

Theorem 1: If n is odd, system (1) has at least one integral straight line. Theorem 2: If m is even, system (1) has at least one integral straight line. Theorem 3: If $(g_1, \cdots, g_n)$ is a set of direction numbers of a critical line of system (1), then $x_1=g_1, \cdots, x_n=g_n$ is a real solution of system (2). The proof on the first theorem is based on the fact that a continuous and nonvanishing vector field defined on an even-dimensional sphere has at least one vector normal to the sphere [Alexandroff and Hopf, Topologie, Berlin, Springer 1935, p. 481]. The proof of theorem 2 depends upon Borsuk's antipodal point theorem [ibid., p. 483]. The proof of theorem 3 uses elementary methods only.

C. S. Coleman (Baltimore, Md.)

6556:

Saito, Tosiya. A note on the linear differential equation of Fuchsian type with algebraic coefficients. Kōdai Math. Sem Rep. **10** (1958), 58–63.

Let $q_1(z), \cdots, q_n(z)$ be algebraic functions on the same closed Riemann surface of genus p, and let

$$y^{(n)} + q_1(z)y^{(n-1)} + \cdots + q_n(z)y = 0$$

be a differential equation of Fuchsian type. The problem treated in this paper is that of determining the number of essentially different equations of this kind with a given number of singularities m. It is shown that, for $p\geq 2$, the number of independent complex parameters entering the equation is $\frac{1}{2}n^2(m+2p-2)+\frac{1}{2}mn+m$. For $p=0$ and $p=1$, the numbers are, respectively, $\frac{1}{2}n^2(m-2)+\frac{1}{2}mn+m-3$ and $\frac{1}{2}mn(n+1)+m-1$.

Z. Nehari (Pittsburgh, Pa.)

6557:

Mikusiński, J. Sur les théorèmes d'unicité et le nombre de solutions linéairement indépendantes. Studia Math. **16** (1957), 95–98.

This note is concerned with purely algebraic aspects of relations between homogeneous differential systems of nth order and systems of n equations of 1st order. Let E be a linear vector space over a (commutative) field of characteristic 0. A linear operator D with domain and range in E may be regarded abstractly as an operation of derivation. If P is a polynomial in one variable, with coefficients from the field, the operator $P(D)$ is defined in an obvious way, and one can consider how many linearly independent solutions there are of $P(D)x=0$, where P is of degree n. Let (I) be the proposition: For every P of degree n, $P(D)x=0$ has at most n linearly independent solutions.

Now consider ordered sets $X=(x_1, \cdots, x_n)$, $x_i\in E$, and an $n\times n$ matrix A of field elements. Let $DX=(Dx_1, \cdots, Dx_n)$. Let (II) be the proposition: For every A the system $DX=AX$ has at most n linearly independent solutions. It is easy to see that (II) implies (I). The author establishes the converse, by using a transformation of matrices to rational canonical form.

If φ is a linear functional on E, $\varphi(x)$ may be thought of as an abstract counterpart of an "initial value" $x(0)$. It is then proved that the uniqueness assertion, "For every P of degree n, if $P(D)x=0$ and if $\varphi(x)=\varphi(Dx)=\cdots=\varphi(D^{n-1}x)=0$, then $x=0$", is logically equivalent to the assertion: "For every A, if $DX=AX$ and $\varphi(x_1)=\cdots=\varphi(x_n)=0$, then $X=(0, \cdots, 0)$". This depends upon the fact, established in a preceding paper [Studia Math. **16** (1957), 41–47; MR **19**, 747], that each x_i here satisfies $P(D)x_i=0$, where P is the characteristic polynomial of A.

A. E. Taylor (Los Angeles, Calif.)

6558:

Mikusiński, J. Sur l'espace linéaire avec dérivation. Studia Math. **16** (1957), 113–123.

The author proceeds further with his algebraic studies of an abstract operation of derivation [cf. the preceding review]. By restricting attention to the space F of all those $x\in E$ such that $P(D)x=0$ for some polynomial P, we can assume that D is a linear operator with domain F and range in F. It is now assumed that, for each P of whatever degree n, $P(D)x=0$ has at most n linearly independent solutions. With this assumption, an investigation is made of when the following proposition Q is true: If $P_k(D)x=0$ has exactly p_k linearly independent solutions ($k=1, 2$), then $P_1(D)P_2(D)x=0$ has exactly p_1+p_2 linearly independent solutions.

It is proved that if Q is true (for all P_1 and P_2) then there exists a linear operator T on F such that $DTx=TDx+x$ for each x. That the existence of such a T is sufficient for the truth of Q was proved in the author's paper cited in the preceding review. Note that T is an abstract generalization of the operation of multiplying $x(t)$ by t, where $Dx(t)=x'(t)$. The proof employs a step-by-step construction, F being arrived at as the linear span of $\bigcup_0^\infty B_n$, where B_n is the set $T(B_{n-1})$ and B_0 is composed of all elements $x_Q, Dx_Q, \cdots, D^{q-1}x_Q$, where x_Q is a nonzero solution of $Q(D)x=0$, Q being irreducible, of degree q; B_0 is generated by considering all Q (of whatever degree) for which an x_Q exists. The definition of T proceeds by induction also.

A. E. Taylor (Los Angeles, Calif.)

6559:

Mikusiński, J. Extensions de l'espace linéaire avec dérivation. Studia Math. **16** (1957), 156–172.

Let C be a commutative field of characteristic 0, and let F be a linear space over C, subject to one or more of the following assumptions, in which D denotes an endomorphism of F.

I: If P is a polynomial of degree n with coefficients from C, the equation $P(D)x=0$ has at most n linearly independent solutions.

II: If $P_1(D)x=0$ and $P_2(D)x=0$ have exactly p_1 and p_2 linearly independent solutions, respectively, then $P_1(D)P_2(D)x=0$ has exactly p_1+p_2 linearly independent solutions.

III: Every element x of F satisfies $P(D)x=0$ for some P.

IV: For each P, of whatever degree p, $P(D)x=0$ has exactly p linearly independent solutions. (This, of course, implies I and II.)

First suppose that F satisfies I and II. If Q is an

irreducible polynomial of degree q and if $Q(D)x=0$ has no nonzero solution in F, the author constructs an extension F^Q of the space F and a corresponding extension of D, such that $Q(D)x=0$ has exactly q linearly independent solutions in F^Q. Moreover, the space F^Q satisfies I and II, and, if F satisfies III, so does F^Q.

A space F with derivation D, if it satisfies III and IV, is determined uniquely up to an isomorphism. If, in addition, C is algebraically closed, then the elements of F can be represented explicitly in the form $\sum_{i=0}^{n} P_i(T)e^{w_iT}$ (n depending on the element). Here T is as in #6558 above; each P_i is a polynomial, and e^{w_iT} is the notation used for a selected nonzero solution of the equation $Dx=w_ix$.

If F satisfies I and II, one can make an adjunction of a "transcendent" element u to obtain a space F^u still satisfying I and II, but such that each element x of F^u-F does not satisfy any equation $P(D)x=0$. In certain realizations of the general theory, the transcendent element corresponds to the Dirac delta function. When F satisfies III and IV and C is algebraically closed, the elements of F^u are of the form $\sum_{i=0}^{n} P_i(T)e^{w_iT}+R(D)u$, where R is a polynomial.

When C is the algebraic closure of the field of generalized functions considered in the author's operational calculus [Studia Math. **11** (1949), 41–70; MR **12**, 189], it determines a space F with derivation satisfying III and IV. In this F, the theory of differential equations with constant coefficients parallels the classical theory, and the operational calculus is given a significant extension.
A. E. Taylor (Los Angeles, Calif.)

6560:

Sikorski, R. On Mikusiński's algebraical theory of differential equations. Studia Math. 16 (1957), 230–236.

In this paper the author gives another proof of the proposition of Mikusiński described in #6558 above — namely, that in a space F with a linear operation D of derivation satisfying certain conditions, there necessarily exists a linear operation T such that $DTx=TDx+x$ for each x. As in Mikusiński's proof the construction of T is by induction, but the details appear to be more simple.
A. E. Taylor (Los Angeles, Calif.)

6561:

Pöschl, Klaus. Über Anwachsen und Nullstellenverteilung der ganzen transzendenten Lösungen linearer Differentialgleichungen. I. J. Reine Angew. Math. 199 (1958), 121–138.

The author discusses the linear differential equation

$$(*)\qquad w^{(m)}(z)+p_1(z)w^{(m-1)}(z)+\cdots+p_m(z)w(z)=0,$$

where the functions $p_k(z)$ are polynomials of the complex variable z. The solutions of (*) are entire functions, and the main concern of this paper is the determination of the order

$$(**)\ \lambda=\limsup_{r\to\infty}\log\log M(r)(\log r)^{-1}\ (M(r)=\max_{|z|=r}|w(z)|)$$

of these solutions. It follows from the properties of (*) that, in the case of a transcendental solution of (*), λ is rational and that (**) can be replaced by

$$\log M(r)=\alpha r^{\lambda}(1+\varepsilon(r)),$$

where α is a finite positive constant and $\varepsilon(r)\to 0$ for $r\to\infty$. The possible values of λ can be characterized in terms of the degree and the coefficient of the highest power of the polynomials $p_k(z)$. The author develops a number of criteria which, in certain cases, make it possible to determine which of these possible orders of growth will actually appear. He also obtains criteria which yield information regarding the distribution of the zeros of solutions of (*).
Z. Nehari (Pittsburgh, Pa.)

6562:

Wintner, Aurel. On a linear differential equation of Briot-Bouquet. Rend. Circ. Mat. Palermo (2) 7 (1958), 42–47.

Let $f(x)$ be a real non-negative function defined for all $x>0$ and having derivatives of all orders. Assume, in addition, that: 1) $Df(x)\geqq 0$;

2) $\int_{+0}^{\infty} f(1/x)e^{-x}dx<\infty$;

3) the function $\varphi(x)=-D^3f(x)$ is totally monotone (i.e., $(-1)^nD^n\varphi(x)\geqq 0$ for $n=0, 1, 2, \cdots$ and $x>0$); and

4) $x^2D^2f(x)\leqq(1+2x)Df(x)$.

Then the differential equation

$$x^2Dy(x)=-y(x)+f(x)$$

has a unique solution $y(x)$ which is totally monotone.
A. G. Azpeitia (Amherst, Mass.)

6563:

Myačin, V. F. On the system of two Briot and Bouquet's equations. Vestnik Leningrad. Univ. 13 (1958), no. 7, 88–102. (Russian. English summary)

Consider the system

$$x(dy_s/dx)=p_{s1}y_1+p_{s2}y_2+F_s(y_1, y_2, x)\quad (s=1, 2),$$

where F_s are power series without constant or linear terms. The following theorem is proved: there are always solutions $y_s(x)\to 0$ when $x\to 0$ with a bounded argument, and these solutions form a one- or two-parameter family if, respectively, one or both characteristic roots of the matrix (p_{ij}) have positive real parts. Certain restrictions in the classical treatment of this subject are thus shown to be inessential. The structure of the series satisfying the system is studied in detail. *J. L. Massera* (Montevideo)

6564:

Putnam, C. R. On the first stability interval of the Hill equation. Quart. Appl. Math. 16 (1958), 421–422.

Assume that λ is real, f is a real-valued, continuous periodic function of period 1, and $\int_0^1 f^+(t)dt\leqq 4$, where $f^+(t)=\max[0, f(t)]$. Let λ_1 be the right end-point of the first stability interval of the Hill equation $x''+(\lambda+f(t))x=0$. It is shown that $\lambda_1\geqq\pi^2(1-\frac{1}{4}\int_0^1 f^+(t)dt)$ is a "best possible" result, in that the assertion becomes false if π^2 is replaced by a larger number or if $\frac{1}{4}$ is replaced by a smaller number.
J. P. LaSalle (Baltimore, Md.)

6565:

Putnam, C. R. On the stability intervals of the Hill equations. J. Soc. Indust. Appl. Math. 7 (1959), 101–106.

Let $f(t)$ be real, continuous and periodic of period p for $(-\infty, +\infty)$. For Hill's equation (*) $x''+(\lambda+f(t))x=0$ there exists a sequence of closed intervals $\mu_n\leqq\lambda\leqq\lambda_n$ ($n=1, 2, \cdots$) — the stability intervals — such that all solutions of (*) are bounded and almost periodic over $(-\infty, +\infty)$ if and only if λ belongs to one of these intervals. In the present paper the author extends a previous result on the first stability interval [#6564 above] and obtains for the right endpoint λ_n of the nth stability interval the inequality

$$\lambda_n^+\geqq\left(\frac{n\pi}{p}\right)^2\left[1-\frac{p}{4n^2}\int_0^p f^+(t)dt\right],$$

where $\lambda_n^+=\max(0, \lambda_n)$, $f^+=\max(0, f(t))$ and the equality holds only if $\lambda_n^+=\lambda_n$ and $f\equiv 0$. This result is shown to be a consequence of the following generalization of Liapounoff's theorem: Suppose that some solution $x(t)\not\equiv 0$ of (*) has at least $n+1$ zeros on $[0, 1]$; then

$$\lambda^+\geq\left(\frac{n\pi}{p}\right)^2\left[1-\frac{p}{4n^2}\int f^+(t)dt\right],$$

where again $\lambda^+=\max(0, \lambda)$ and the equality holds only if $\lambda^+=\lambda$ and $f\equiv 0$. The proof of this latter theorem uses arguments of P. Hartman [Amer. J. Math. **71** (1949), 71–79; **73** (1951), 955–962; MR **10**, 455; **13**, 652] and P. Hartman and A. Wintner [ibid. **73** (1951), 885–890; MR **13**, 652]. *W. C. Rheinboldt* (Washington, D.C.)

6566:

Olech, C. On the characteristic exponents of the second order linear ordinary differential equation. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 573–579.

Let the characteristic exponent, λ_0, of a function $x(t)$ be defined by

$$\lambda_0=\limsup_{t\to\infty} t^{-1}\log|x(t)|.$$

The author studies the characteristic exponents of nontrivial solutions of (1) $x''+a(t)x=0$ $('=d/dt)$, where $a(t)$ is positive, piecewise continuous, and bounded: $0<\alpha^2\leq a(t)\leq\beta^2$. He obtains best-possible upper and lower estimates for the characteristic exponents of all nontrivial solutions of (1) obtainable from knowledge of just α and β. He shows that $-\rho_2\leq\lambda_0\leq\rho_2$ for every nontrivial solution of (1), where ρ_2 is the unique nonnegative root of

$$(\rho^2+\beta^2)^{\frac12}(\rho^2+\alpha^2)^{-\frac12}\times\exp[-\rho\alpha^{-1}\cot^{-1}\rho\alpha^{-1}-\rho\beta^{-1}\cot^{-1}(-\rho\beta^{-1})]=1.$$

He shows further that there exists a piecewise constant function $\bar{a}(t)$ taking on just the values α^2 and β^2 for which (1) has nontrivial solutions, one with characteristic exponent $-\rho_2$ and another with characteristic exponent $+\rho_2$. As a corollary he shows that all solutions of (3) $y''+y'+b(t)y=0$, $\alpha^2+\frac14\leq b(t)\leq\beta^2+\frac14$, approach zero exponentially if $\rho_2<\frac12$, while if $\rho_2>\frac12$, (3) may have unbounded solutions.

The case in which β is replaced by $+\infty$ is considered, and an example is given in which (1) has solutions with unbounded characteristic exponents. The same result is obtained for (3) in this case. The existence in both cases of unbounded solutions of (3) contradicts an assertion of Wintner [J. Math. Mech. **6** (1957), 109–117; MR **18**, 737]. *W. S. Loud* (Minneapolis, Minn.)

6567:

Takahashi, Shin-ichi. Some boundedness theorems of solutions of linear differential equations. Proc. Japan Acad. **34** (1958), 599–603.

In the differential equation

$$x''+p(t)x'+q(t)x=f(t),$$

let $p(t)$, $q(t)$ and $f(t)$ be real-valued, continuous functions on the half-line $I: c\leq t<\infty$. It is shown that all solutions of the equation are bounded on I if there exists a differentiable positive function $\lambda(t)$ on I satisfying

$$\int_0^\infty\lambda(t)^{-\frac12}|\lambda(t)-q(t)|dt<\infty,\quad\int_0^\infty\lambda(t)^{-\frac12}|f(t)|dt<\infty,$$

and either

$$\int_0^\infty|2p(t)+\lambda'(t)/\lambda(t)|dt<\infty$$

or the condition

$$2p(t)\lambda(t)+\lambda'(t)\geq 0.$$

Various consequences of this result, in particular if $\lambda(t)\equiv a>0$ or $\lambda(t)\equiv q(t)$ (in case $q(t)>0$), are noted. Linear inhomogeneous systems are also considered. *C. R. Putnam* (Lafayette, Ind.)

6568:

Smart, D. R. Eigenfunction expansions in L^p and C. Illinois J. Math. **3** (1959), 82–97.

The author extends the abstract-operator perturbation theory approach to eigenfunction expansion theory, obtaining results on the conditional convergence of eigenfunction expansions in the spaces L_p and C. The principal result which makes this possible is the following: if T is an unbounded linear operator in a B-space, the spectrum of T being discrete and the corresponding projections being suitably restricted, and if S is a suitably restricted perturbation, then we have $\lim_{n\to\infty}\|E_n(T)-E_n(T+S)\|=0$, $E_n(A)$ denoting the projection corresponding to the n eigenvalues of smallest modulus of the operator A. *J. T. Schwartz* (New York, N.Y.)

6569:

Pöschl, Klaus. Über Anwachsen und Nullstellenverteilung der ganzen transzendenten Lösungen linearer Differentialgleichungen. II. J. Reine Angew. Math. **200** (1958), 129–139.

The author seeks to determine the cases in which Hille's well-known results concerning the growth and distribution of zeros of the solutions of second-order linear differential equations with entire coefficients [Trans. Amer. Math. Soc. **23** (1922), 350–385] can be generalized to higher-order equations. Hille's work is based on an equivalent integral equation which is obtained via a Liouville transformation of the original differential equation. This procedure cannot, in general, be applied to higher-order equations, since the freedom of manipulation provided by the two arbitrary functions entering the Liouville transformation is not sufficient for casting the equation into the desired form. Nevertheless, there are some cases where this is possible, and these cases are considered in detail for third and fourth order equations. *Z. Nehari* (Pittsburgh, Pa.)

6570:

Mitropol'skiĭ, Yu. A. Asymptotic methods of N. M. Krylov and N. N. Bogolyubov and their further development. Rev. Math. Pures Appl. **1** (1956), no. 3, 15–26. (Russian)

A general lecture, with many references to recent contributions and applications, of the general asymptotic method initiated and strongly impulsed by the late N. M. Krylov and by N. N. Bogoliubov. *S. Lefschetz* (Mexico, D.F.)

6571:

Sibuya, Yasutaka. Remarques sur la théorie des centres aux dimensions supérieures. J. Math. Soc. Japan **8** (1956), 1–6.

6572:

Glatenok, I. V. Foundation of the method of harmonic balance. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **1958**, no. 1, 39–52. (Russian)

The method of harmonic balance yields a first approximation of the form $y=a\sin\omega t$ (a, ω constants) to the periodic solution of $\ddot{y}=f(y, \dot{y})$. The author derives a

set of sufficient conditions on $f(y, \dot{y})$ under which $\ddot{y}=f(y, \dot{y})$ has a stable periodic solution whose amplitude A differs by less than a known amount from a. The conditions are too involved for reproduction, but in essence they insure that the Fourier coefficients of $f(y, \dot{y})$ are sufficiently small. *L. A. Zadeh* (New York, N.Y.)

6573:

Čžao, Ši-Guan. On conditional stability of saddle systems of ordinary differential equations in the critical case. Sci. Record (N.S.) **1** (1957), 301–305. (Russian)

This is an imperfect résumé of a Chinese paper: Some theorems on stability according to the first approximation [Mém. People's Univ. North East China. Nat. Sci. Ser. no. 1 (1956)]. The paper deals with (1) $\dot{y}=Y(y)$ ($(n+k)$-vectors), where $Y(0)=0$, Y is of class C^1 in some neighborhood $\|y\|<h$, and at the origin there are k zero characteristic roots, m with negative and $n-m$ with positive real parts. Let the system be reduced to (2) $\dot{x}=X(x, z)$, $\dot{z}=Z(x, z)$, where x is a k-vector, z an n-vector, the nonzero roots being those of the Jacobian $\partial Z/\partial z$ at the origin. The auxiliary system (3) $\dot{x}=X(x, z(t))$ is considered, where $z(t)$ is arbitrary (and probably small). Stability of (3) is defined in the obvious way. Theorem 1: If (3) is stable regardless of $z(t)$ (small enough?), if for two functions $z(t)$, $z'(t)$ and corresponding solutions $x(t)$, $x'(t)$ of (3) we have a Z-Lipschitz condition with small constant, then (2) has a manifold of stable solutions depending upon $m+k$ parameters.

Consider now the non-autonomous system (4) $\dot{x}=X(t, x, z)$, $\dot{z}=P(t)z+Z(t, x, z)$, and let (5) represent the same system but with $Z=0$. Theorem 2: If (a) the system (5) has a solution such that

$$|z_{si}(t, t_0)|<Me^{-\lambda_i(t-t_0)},\ i=1, 2, \cdots, m,$$
$$|z_{sj}(t, t_0)|<Me^{\lambda_j(t-t_0)},\ j=m+1, \cdots, n,$$

where λ_i and λ_j are both positive; (b) the origin is stable for $\dot{x}=X$ regardless of choice of $z(t)$; (c) Z satisfies a Lipschitz condition uniformly in t; then (4) has a amily of bounded solutions depending upon $m+k$ arbitrary constants. *S. Lefschetz* (Mexico, D.F.)

6574:

Gel'man, A. E. A test for the existence of certain classes of solutions of a non-linear differential equation and some estimates by the small parameter method. Dokl. Akad. Nauk SSSR (N.S.) **118** (1958), 1063–1065. (Russian)

In articles of A. A. Kruming [Ukrain Mat. Ž. **5** (1953), 434–438; MR **15**, 624] and D. C. Lewis [Duke Math. J. **22** (1955), 39–56; MR **17**, 38] the question is studied of estimating the region of variability of a parameter for which a periodic solution of a non-linear differential equation exists (or also is expansible in a series in this parameter). The methods used in these articles cannot on principle be employed for the solution of analogous problems in the case of a quasi-periodic or almost-periodic solution. The ideas developed in the present article may be used to solve this problem not only in the periodic case but also for a quasi-periodic or an almost-periodic solution and for a solution which is bounded on the whole real axis. *From the introduction*

6575:

Chin, Yuan-shun. On the equivalence problem of differential equations and difference-differential equations in the theory of stability. Sci. Record (N.S.) **1** (1957), 287–289.

Consider the differential-difference equation $u'(t)=pu(t)+qu(t-d)+g(u(t), u(t-d))$. The author shows that if $p+q<0$, then for d sufficiently small the solutions of the linear approximation are representative of the solution of the original equation, provided that g is a quadratically nonlinear function. The problem arises in the consideration of control processes where small time-lags occur.

R. Bellman (Santa Monica, Calif.)

PARTIAL DIFFERENTIAL EQUATIONS

See also 6806, 6807, 6828, 6856.

6576:

✱Krzyżański, Mirosław. Równania różniczkowe cząstkowe rzędu drugiego. [Partial differential equations of second order.] Biblioteka Matematyczna, Tom 15. Państwowe Wydawnictwo Naukowe, Warsaw, 1957. 617 pp. zł 52.

There are 9 chapters, with the headings: Cauchy problem for analytic functions; characteristics and classification of partial differential equations of second order; boundary problems, their types and their connection with the problems of physics; linear equations; fundamental formula; uniqueness of solutions of boundary problems for equations of second order, continuity with respect to boundary conditions; Laplace equations, harmonic functions; elements of potential theory, applications to the theory of harmonic functions, Poisson equations; linear elliptic equations; normal parabolic equations.

6577:

Ilyin, V. A. On the uniform convergence of expansions in characteristic functions when the sum is taken in the order of increasing characteristic numbers. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 698–701. (Russian)

Let G be a smoothly bounded domain in an Euclidean space E^n of even dimension, $u_i(x)$ an orthonormal family of eigenfunctions of the Laplace operator on G under homogeneous boundary conditions corresponding to either the Dirichlet, Neumann, or Robin's problem. If f is an arbitrary function in $L^2(G)$, it has an expansion of the form $f=\sum_i f_i u_i(x)$, where the sum is written with monotone order of the corresponding characteristic values λ_i for the eigenfunctions u_i. The author states and sketches the proof of a number of results centering around the problem of giving conditions on f which are sufficient for the uniform convergence of the above series on compact subsets of G. His principal result is the following: Suppose that both of the two following conditions are satisfied: (1) $f\in W^{\frac{1}{2}n,p}(G)$ ($p>2$), i.e., all derivatives of f in the distribution sense for orders $\leq\frac{1}{2}n$ lie in $L^p(G)$, and (2) f and all its Laplacians of order $\leq\frac{1}{2}(n-2)$ in the case of the Dirichlet problem, or of order $\leq\frac{1}{2}(n-4)$ in the case of the Neumann and Robin's problems, satisfy the corresponding homogeneous boundary condition (in a suitable extended sense). Then the eigenfunction expansion series for f converges uniformly on every interior subdomain of G, and so, moreover, does the series $\sum_i f_i\lambda_i^\alpha u_i(x)$, where α is any number satisfying the inequality $\alpha<n(p-2)/4p$ for $2<p<2n/(n-1)$, or $\alpha<\frac{1}{4}$ for $p\geq 2n/(n-1)$. *F. Browder* (New Haven, Conn.)

6578:

Gluško, V. P.; and Kreĭn, S. G. Fractional powers of differential operators and imbedding theorems. Dokl. Akad. Nauk SSSR 122 (1958), 963–966. (Russian)

Suppose G is a bounded region in n-dimensional Euclidean space ($n\geqq 2$), star with respect to all points of some sphere contained in it. Let A be a positive definite self-adjoint operator in $L_2(G)$ which is produced by a differential operator of even order with a system of homogeneous boundary conditions.

The operator A is called strongly invertible if

$$\|A^{-1}f\|_{W_2^l}\leqq C\|f\|_2 \quad (f\in L_2),$$

where $\|\ \|_{W_2^l}$ is the norm in the space of Sobolev, W_2^l [Nekotorye primeneniya funkcional'nogo analiza v matematičeskoĭ fizike, Izdat. Leningrad. Gos. Univ., Leningrad, 1950; MR 14, 565]. The authors are interested in the question of into which space the operator $A^{-\alpha}$, $0<\alpha<1$, carries $L_2(G)$.

Theorem 1. Suppose A is strongly invertible, $0<\alpha<1$ and $r=\gamma l-n/2$; the following cases occur. a) r is positive and not an integer. Then $A^{-\alpha}$ is a completely continuous operator acting from L_2 to $C_{m,\nu}$ — the space of functions having partial derivatives of order $m=[r]$ satisfying a Hölder condition with exponent $\nu<r-[r]$. b) r is a positive integer. Then $A^{-\alpha}$ is a completely continuous operator acting from L_2 into $C_{m,\nu}$, where $m=r-1$ and $\nu<1$. c) $r\leqq 0$. Then $A^{-\alpha}$ is a completely continuous operator acting from L_2 into L_q, where $q<-n/r$.

Let D^m be some partial derivative of order m and $D_h^m f(P)=|M-P|^{-h}D^m f(P)$, where $h\geqq 0$ and M is some point of $\bar{G}$.

Theorem 2. If A is strongly invertible and $\gamma l-n/2\leqq m<\gamma l$, then $D^mA^{-\alpha}$ is a completely continuous operator acting from L_p to L_q, where $1/q>\frac{1}{2}-(\gamma l-m)/n$.

Further theorems give estimates on the norm of D_h^m acting in the space L_q and estimates on the norm of $D_h^mA^{-\alpha}$ acting in L_2 which are independent of M. An estimate is also given on the uniform norm of an expression of the form

$$|D^m\varphi(P)-D^m\varphi(Q)|/|P-Q|^\nu.$$

The proofs of these results depend on theorems of O. A. Ladyženskaya [Dokl. Akad. Nauk SSSR **75** (1950), 765–768; MR **12**, 615], O. V. Gyseva [ibid. **102** (1950), 1069–1072; MR **17**, 161], S. L. Sobolev [loc. cit.], S. G. Kreĭn and P. E. Sobolevskiĭ [Dokl. Akad. Nauk SSSR **118** (1958), 233–236; MR **20** #6043], and L. Nirenberg [Comm. Pure Appl. Math. **9** (1956), 509–529; MR **19**, 962].

A. Devinatz (St. Louis, Mo.)

6579:

Krasnosel'skiĭ, M. A.; and Pustyl'nik, E. I. The use of the fractional powers of operators in studying Fourier series of eigenfunctions of differential operators. Dokl. Akad. Nauk SSSR 122 (1958), 978–981. (Russian)

During the past few years there has been a heightened interest in the investigation of Fourier series of eigenfunctions of differential operators. In this report the authors attempt to put these investigations on an abstract operator-theoretic basis.

Let T be a positive definite self-adjoint operator in a Hilbert space H with a completely continuous inverse having eigenvectors u_k. Let Ω_α be the domain of the operator T^α, $\alpha>0$. The first theorem states that if $T^{-\beta}$ is a continuous operator acting from H to some space $E\subset H$ (E has, in general, a different norm than H), then the series $\sum_1^\infty (f, u_k)u_k$ converges to f, in the norm of E, for all $f\in\Omega_{\beta+\alpha}$ ($\alpha\geqq 0$). An estimate is also given on the rapidity of convergence.

Several other theorems give information on the range of an operator A^α, $0<\alpha<1$, under various conditions. We cite a representative case: Suppose A is a positive definite self-adjoint operator in $L^2(G)$ (G a bounded closed domain in n-dimensional Euclidean space) which takes L^2 into C (continuous functions — usual norm) and is continuous. Then A^α, $0<\alpha<1$, takes L^2 into L^p, where $p\leqq 2/(1-\alpha)$. If A is completely continuous, then A^α is also, when acting from L^2 to L^p.

Conditions are also given under which a generalized Fourier series may be differentiated termwise, the differentiation being in the sense of S. L. Sobolev [Nekotorye primeneniya funkcional'nogo analiza v matematičeskoĭ fizike, Izdat. Leningrad. Gos. Univ., Leningrad, 1950; MR **14**, 565].

A. Devinatz (St. Louis, Mo.)

6580:

Haimovici, M. Systèmes Pfaff du II-e genre, à caractère réductible. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Şti. Ser. I. 6 (1955), no. 3–4, 17–25. (Romanian. Russian and French summaries)

"In this note the author considers a Pfaff system of genus II and kind I. Sufficient conditions for the reduction of the character of the system are studied together with the method of effectively reducing the character when these conditions are satisfied. In the cases in which this character reduces to one, the integration reduces to that of a system of ordinary differential equations." (From the introduction of the author)

R. Blum (Saskatoon, Sask.)

6581:

Haimovici, M. Systèmes différentiels à caractère réductible. Com. Acad. R. P. Romîne 6 (1956), 975–980. (Rom nian. Russian and French summaries)

In a previous paper [#6580] the author considered differential systems of genus 2, and proved a result which constitutes a generalization of the method of Darboux for integrating partial differential equations of the second order in two independent variables. The present note announces the corresponding result for systems of genus p, which is to be published in detail later.

J. B. Diaz (College Park, Md.)

6582:

Power, G.; and Jackson, H. L. W. Certain two-dimensional solutions to Poisson's equation. Appl. Sci. Res. B 7 (1958), 249–256.

The author considers the following problem. Find functions $\phi_1(x, y)$ and $\phi_2(x, y)$ such that

$$\nabla^2\phi_1=f_1(x, y) \text{ in } |z|<1,$$
$$\nabla^2\phi_2=f_2(x, y) \text{ in } |z|>1,$$
$$\phi_1=\phi_2,\quad k_1\frac{\partial\phi_1}{\partial r}=k_2\frac{\partial\phi_2}{\partial r} \text{ on } |z|=1,$$
$$z=x+iy.$$

The problem is reduced to the determination of four analytic functions of z, use being made of conditions at the origin and at infinity. Two examples are given and some remarks made concerning the extension to non-circular boundaries by means of conformal mapping.

R. C. MacCamy (Pittsburgh, Pa.)

6583:

Pettineo, Benedetto. Sul prolungamento analitico delle soluzioni di talune equazioni a derivate parziali della Fisica-Matematica. Atti Accad. Sci. Lett. Arti Palermo. Parte I (4) **16** (1955/56), 27–33 (1957).

L'A. considera l'equazione dell'elasticità

$$\Delta u + k \text{ grad div } u = 0 \quad (-1 < k < +\infty)$$

e dimostra che ogni soluzione u, continua in un dominio T, è prolungabile analiticamente attraverso ogni porzione analitica Σ della frontiera FT, qualora assuma valori analitici su Σ. La dimostrazione si fonda su precedenti risultati dell'A. [Ann. Mat. Pura Appl. (4) **41** (1956), 221–255; MR **19**, 553]. *L. Amerio* (Milan)

6584:

Leis, Rolf. Über das Neumannsche Randwertproblem für die Helmholtzsche Schwingungsgleichung. Arch. Rational Mech. Anal. **2** (1958), 101–113.

Let G be the exterior of a three-dimensional bounded domain with smooth boundary Ω, and consider the second boundary value problem: (1) $\Delta U + k^2 U = 0$ in G, k const. $\neq 0$, Im $k \geqq 0$; (2) $\partial U/\partial n + \delta U = \gamma$ on Ω, where δ, γ are continuous functions and $\delta \leqq 0$ if Im $k \neq 0$; (3) as $r \to \infty$, $\partial U/\partial r = ikU + o(1/r)$. The author proves existence and uniqueness of a solution of the system (1)–(3). It is represented in terms of simple and double layers. The methods used are similar to those of H. Weyl [Math. Z. **55** (1952), 187–198; MR **14**, 225] and C. Müller [Math. Z. **56** (1952), 80–83; MR **14**, 518]. Use is also made of a uniqueness theorem of F. Rellich [Jber. Deutsch. Math. Verein. **53** (1943), 57–65; MR **8**, 204]. *A. Friedman* (Berkeley, Calif.)

6585:

Oleĭnik, O. A. Partial differential equations with a small parameter in the highest derivatives. Colloq. Math. **5** (1958), 216–222. (Russian)

The paper surveys some of the results which were obtained by N. Levinson [Ann. of Math. (2) **51** (1950), 428–445; MR **11**, 439], S. L. Kamenomostskaya [Mat. Sb. N.S. 31(73) (1952), 703–708; MR **14**, 877] and the author [ibid. 31(73) (1952), 104–117; MR **14**, 560]. For further results on related questions see D. G. Aronson, J. Rational Mech. Anal. **5** (1956), 1003–1014 [MR **19**, 557]. *A. Friedman* (Berkeley, Calif.)

6586:

Tong, Kwang-chang. On singular Cauchy problems of hyperbolic partial differential equations. Sci. Record (N.S.) **1** (1957), 319–322.

The Cauchy problem for the hyperbolic equation $Lu \equiv \sum a_{ij} u_{x_i x_j} + b_i u_{x_i} + cu = f$ is called singular of the first kind if the initial values are prescribed on a surface where one or more of the coefficients become infinite. The problem is called singular of the second kind if the region of hyperbolicity degenerates on the surface of initial values. The author announces a solution of the singular problems of the first and second kind under appropriate restrictions on the behavior of the coefficients in a neighborhood of the initial surface. For problems of the first kind this extends a result of Krasnov [Dokl. Akad. Nauk SSSR **107** (1956), 789–792; MR **19**, 748]. For problems of the second kind there are two theorems; one extends a result of Karapetyan [ibid. **106** (1956), 963–966; MR **19**, 748], the other extends to n variables (under somewhat different hypotheses) a result of the reviewer [Canad. J. Math. **6** (1954), 542–553; MR **16**, 255] for two independent variables. It is stated that the methods are based on the results of Sobolev [Mat. Sb. N.S. **11** (1942), 155–200; MR **5**, 8]. *M. H. Protter* (Berkeley, Calif.)

6587:

Majcher, G. Sur un problème mixte pour l'équation du type hyperbolique. Ann. Polon. Math. **5** (1958), 121–133.

Consider the hyperbolic operator

$$H[u] \equiv u_{xy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u$$

defined on the domain D which is bounded by the half line $y=0$, $x \geqq 0$, by the line $x = x_0 > 0$, and by a curve Γ issuing from the origin and given by $x = \theta(y)$, $y \geqq 0$, where $\theta'(y) > 0$. The paper establishes the existence and uniqueness of a solution u of $H[u] = \lambda f(x, y)$ such that $u(x, 0) = \nu h(x)$ and $u_y = \mu\phi(y, u, u_x)$ on Γ. Here λ, ν, μ are parameters which in general must be restricted in range; the function ϕ satisfies a Lipschitz condition in u and u_x, and the coefficients and boundary functions are subjected to a variety of restrictions. In addition to this general problem with a non-linear boundary condition, the author investigates the special problem where the condition on Γ is linear:

$$A(y)u_x + B(y)u_y + C(y)u = g(y).$$

It is not required that this last equation be solvable for u_x or u_y. The boundary problems are handled by writing down a known general solution of the differential equation, involving a pair of arbitrary functions. The boundary conditions then become (non-linear) integral equations involving this pair of functions. The main work of the paper is an investigation of these integral equations, employing the classical tools of the resolvent kernel and Picard's iteration. *R. W. McKelvey* (Boulder, Colo.)

6588:

Satō, Tokui. Sur le problème de Neumann pour l'équation $\Delta u(P) = F(P, u(P), \partial u(P))$. Proc. Japan Acad. **34** (1958), 107–109.

Sia T un dominio "regolare" dello spazio a tre dimensioni ed S la frontiera di T, $f(P)$ una funzione continua su S, e $F(x_1, x_2, x_3; u; p_1, p_2, p_3)$ una funzione continua e limitata per $(x_1, x_2, x_3) \in S \cup T$, $-\infty < u, p_1, p_2, p_3 < +\infty$. Esiste almeno una funzione continua con le derivate parziali prime in $S \cup T$ che soddisfa in T, in un senso generalizzato, l'equazione $\Delta u(P) = F(P, u(P), \partial u(P))$ e su S la condizione $du/dn = f(P) + C$, essendo C una costante opportuna. *G. Stampacchia* (Genoa)

6589:

Murakami, Haruo. On non-linear partial differential equations of parabolic types. I, II, III. Proc. Japan Acad. 33 (1957), 530–535, 616–621, 622–627.

L'operatore $\mathfrak{M}$ generalizza l'operatore del calore $\partial^2/\partial x^2 - \partial/\partial y$ in un senso già introdotto da B. Pini [Rend. Sem. Mat. Univ. Padova **27** (1957), 149–161; MR **19**, 1060]. Le equazioni considerate sono del tipo: (E$_1$) $\mathfrak{M}u = f(x, y, u)$; (E$_2$) $\mathfrak{M}u = f(x, y, u, \partial u/\partial x)$; (E$_3$) $\mathfrak{M}u = f(x, y, u, \partial u/\partial x, \partial u/\partial y)$.

I domini D sono del tipo $a \leqq y \leqq b$, $\lambda_1(y) \leqq x \leqq \lambda_2(y)$, e con S si indica il segmento aperto superiore della frontiera ∂D di D e con C l'insieme $\partial D - S$. Le soluzioni di (E$_1$), (E$_2$), (E$_3$) sono funzioni continue in $D - C$ che ammettono le derivate che compaiono nei secondi membri di (E$_1$), (E$_2$), (E$_3$), rispettivamente. Sono dimostrati alcuni teoremi di confronto per le soluzioni delle equazioni (E$_3$) dai quali sono dedotte alcune formole di maggiorazione per le

soluzioni di (E_2). Alcuni teoremi di unicità e di esistenza — che generalizzano quelli di B. Pini [loc. cit.] — sono dimostrati per le soluzioni dell'equazione (E_2) che si annullano su C. Fra i teoremi di esistenza riportiamo il seguente. "Le funzioni $\lambda_1(y)$, $\lambda_2(y)$, che definiscono il dominio D, siano continue con le derivate prime; la funzione $f(x, y, u, p)$, definita per $(x, y) \in D$ e $-\infty<u, p<+\infty$, sia limitata in ogni sottoinsieme per cui $-\infty<-h\leq u\leq h<+\infty$ e soddisfi la condizione: $f(x, y, u_1, p)-f(x, y, u_2, p)> -k(u_1-u_2)$ per $(x, y) \in D$ e $u_1>u_2$ (k costante). Allora (E_2) ammette almeno una soluzione che è continua in $\bar{D}=D+C+S$ e che si annulla su C."

Teoremi analoghi ai teoremi di Harnack per le funzioni armoniche sono provati per le soluzioni di (E_2). Risultati più precisi sono poi dati nel caso dell'equazione (E_1) introducendo la nozione di barriera di (E_1) rispetto ad una funzione in un punto di C. Alcuni cenni all'estensione al caso pluridimensionale completano il lavoro.

G. Stampacchia (Genoa)

6590:

Murakami, Haruo. On the regularity of domains for parabolic equations. Proc. Japan Acad. **34** (1958), 347–348.

Sia G un dominio dello spazio ad m dimensioni e Γ il suo contorno, e sia $D=G\times(0, \infty)$, $B=\Gamma\times[0, \infty)$. Si dimostra che, se il dominio G è regolare per l'equazione di Laplace, il corrispondente dominio D ammette in ogni punto di $G\cup B$ una barriera nel senso già introdotto [#6589], e ciò implica che il primo problema al contorno per l'equazione $\mathfrak{M}u=f(x, t, u)$, che generalizza l'equazione del calore, è risolubile in un senso opportuno qualunque siano i dati continui su $G\cup B$. *G. Stampacchia* (Genoa)

6591:

Murakami, Haruo. Relations between solutions of parabolic and elliptic differential equations. Proc. Japan Acad. 34 (1958), 349–352.

L'Autore studia in quali condizioni una soluzione generalizzata (nel senso considerato in #6589) del primo problema al contorno per l'equazione $\sum_{i=1}^m \partial^2u/\partial x_i^2-\partial u/\partial t= f(x, t, u)$ tende per $t\to\infty$ ad una soluzione generalizzata del problema di Dirichlet per l'equazione $\sum_{i=1}^m \partial^2v/\partial x_i^2= f(x, v)$, in un senso considerato da T. Satō [Compositio Math. 12 (1954), 157–177; MR 17, 474].

G. Stampacchia (Genoa)

6592:

Nash, J. Continuity of solutions of parabolic and elliptic equations. Amer. J. Math. **80** (1958), 931–954.

In this paper, the writer considers bounded solutions $T(x, t)$ $(x=x^1, \cdots, x^n)$ of parabolic equations of the form $T_t=\nabla\cdot a\cdot\nabla T$, in which the eigenvalues of the matrix $a=\|a_{ij}(x, t)\|$ are always between two numbers c_1 and c_2, with $0<c_1<c_2$, and ∇ denotes the x-gradient. He first proves that any solution $T(x, t)$ such that $|T(x, t)|\leq B$ for $t>t_0$ satisfies a Hölder condition of the form

$$|T(x_1, t)-T(x_2, t)|\leq AB(|x_1-x_2|/(t-t_0)^{\frac{1}{2}})^\alpha,\quad 0<\alpha<1,$$

where A and α depend only on n, c_1, c_2; he also shows that any such T satisfies a corresponding Hölder condition in the time variable. The writer bases his proof on a succession of inequalities for fundamental solutions $S(x, t; \bar{x}, \bar{t})$ which are solutions for $t>\bar{t}$ and have a unit source at $(\bar{x}, \bar{t})$. Using these results he shows that bounded solutions of the elliptic equations $\nabla\cdot a\cdot\nabla T=0$ satisfy certain Hölder conditions on interior domains which depend only on sup $|T|$, n, c_1, c_2, and the distance of the interior domain from the boundary of the domain of definition. This result generalizes to the case $n>2$ an old result of the reviewer [Trans. Amer. Math. Soc. **43** (1938), 126–166], which he used to prove the differentiability of the solutions of certain variational problems and which Nirenberg [Comm. Pure Appl. Math. **6** (1953), 103–156, 395; MR **16**, 367] used to establish the existence of the solutions of certain quasilinear elliptic equations. Recently, E. DeGiorgi has proved (by completely different methods) similar results in the elliptic case only [Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3) **3** (1957), 25–43; MR **20**#172], but for solutions which may only be in L_2. The present writer also announces results concerning continuity on the boundary in the elliptic case. *C. B. Morrey, Jr.* (Berkeley, Calif.)

6593:

Aleksandryan, R. A. On the correctness of a mixed problem and on the spectral equivalence of two operators associated with it. Izv. Akad. Nauk Armyan. SSR Ser. Fiz.-Mat. Nauk **10** (1957), no. 1, 69–83. (Russian. Armenian summary)

In a previous paper the author has summarised results concerning the system

$$\partial^2V_x/\partial t^2=-V_x+\partial P/\partial x,\quad \partial^2V_y/\partial t^2=\partial P/\partial y,$$

subject to div $V=\partial V_x/\partial x+\partial V_y/\partial y=0$ and $P=0$ on a curve Γ, related equations being an operator equivalent $\partial^2V/\partial t^2=AV$ and the scalar equation $\partial^2\Delta u/\partial t^2+\partial^2u/\partial y^2=0$ [Dokl. Akad. Nauk SSSR **73** (1950), 631–634; MR **12**, 615]. He now gives a detailed exposition, covering also an operator version $\partial^2u/\partial t^2=Bu$ of the last equation; here $B=-\Delta^{-1}\partial^2/\partial y^2$, subject to the null boundary condition on Γ. A and B are shown to have the same spectra.

F. V. Atkinson (Canberra City)

6594:

Drapkin, A. B. Asymptotic expressions for eigenvalues and characteristic functions of a class of elliptical systems. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 465–467. (Russian)

The writer states some results concerning the asymptotic distribution of eigenvalues and eigenfunctions for the Dirichlet problem for the class of strongly elliptic, positive definite systems of second-order on a reasonably bounded domain of three-dimensional Euclidean space. A proof is sketched, which applies the Tauberian argument of Carleman [Ber. Verh. Sachs. Akad. Wiss. Leipzig **88** (1936), 119–132] together with results of Ya. B. Lopatinski on fundamental solutions and Green's functions for such systems. *F. Browder* (New Haven, Conn.)

6595:

Smirnov, M. M. The first problem for a hyperbolic equation of the fourth order. Vestnik Leningrad. Univ. **13** (1958), no. 19, 55–57. (Russian. English summary)

"This paper treats the problem of finding a solution $u=u(x,y)$ of the equation $\partial^4u/\partial x^4-2\partial^4u/\partial x^2\partial y^2+\partial^4u/\partial y^4=0$ in the plane domain D:$0\leq x<1, 0\geq y\geq -\gamma(x)$, $\gamma(x)>0$ and satisfying the boundary conditions $u|_{y=0}=\phi_0(x)$, $\partial u/\partial y|_{y=0}=\phi_1(x)$, $u|_{y=-\gamma(x)}=\psi_0(x)$, $\partial u/\partial y|_{y=-\gamma(x)}=\psi_1(x)$, $0\leq x\leq 1$. It is proved that there exists a continuum of solutions of this problem." *Author's summary*

6596:

U, Čžun-haĭ. The Cauchy problem for some degenerate hyperbolic 4th-degree equations. Sci. Record (N.S.) **3** (1959), 49–54. (Russian)

It is required to find a solution $u(x, y)$ for $y>0$ and for

non-negative m, n of the equation

$$\left(y^m\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)\left(y^n\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}\right)=0$$

satisfying on the segment $a\leqq x\leqq b$ the conditions $u=\varphi_0(x)$, $u_y=\varphi_1(x)$, $u_{yy}=\varphi_2(x)$, $u_{yyy}=\varphi_3(x)$. It is proved that the problem is correctly set under any one of the following combinations of conditions: (1) for $m\geqq 0$, $n=0$, $\varphi_0\in C^4$, $\varphi_1\in C^4$, $\varphi_2\in C^2$, $\varphi_3\in C^2$; (2) for $m\geqq 0$, $n=1$, $\varphi_0\in C^4$, $\varphi_i\in C^2$ $(i=1, 2, 3)$; (3) for $m\geqq 0$, $n>1$, $\varphi_i\in C^2$ $(i=0, 1, 2, 3)$; and (4) for $m\geqq 0$, $0<n<1$, $\varphi_0\equiv ax+b$ (a, b arbitrary constants, different from a, b above), $\varphi_i\in C^2$ $(i=1, 2, 3)$. In each of the four cases the solution is exhibited in terms of solutions of two Cauchy problems for second-order equations, which in turn are solved explicitly by formula.

R. N. Goss (San Diego, Calif.)

6597:

Zagorskiĭ, T. Ya. Some mixed problems for parabolic systems of differential equations. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 359–362. (Russian)

In this article the mixed problem for a parabolic system is reduced to a solvable integral equation. As was done by Z. Ya. Šapiro [Mat. Sb. (N.S.) **28(70)** (1951), 55–78; MR **14**, 652] and Ya. B. Lopatinskiĭ [Ukrain. Mat. Ž. **5** (1953), 123–151; MR **17**, 494] for elliptic systems, we here use as kernel the Green's function earlier constructed by us [same Dokl. **106** (1956), 11–14; MR **17**, 857] for the mixed problem for a half-space. *From the introduction*

6598:

Maurin, Lidia. Über die Fouriersche Lösung von gemischten Problemen in beliebigen Gebieten für eine gewisse Klasse von inhomogenen Differentialgleichungssystemen mit partiellen Ableitungen. Studia Math. **16** (1957), 200–229.

Let Ω be a not necessarily bounded domain of E^n, and consider the system of equations (1): $\partial^k u_i(x, t)/\partial t^k = -(A_x u)_i(x, t)+f_i(x, t)$ ($k=1$ or 2; $i=1, 2, \cdots, r$), where $x=(x_1, \cdots, x_n)$ and $0\leqq t\leqq T$. It is assumed that A_x is a formally self-adjoint elliptic system of r differential operators of order σ such that the restriction A_0 of $A=A_x$ to the C^σ-vector functions $u(x)=(u_1(x), \cdots, u_r(x))$ with compact supports in Ω admits, in the space $L^{2,r}(\Omega)$ of square integrable vector functions $u(x)$, a self-adjoint extension A_1 which is bounded from below, i.e., $(A_1u, u)\geqq m_0(u, u)$ for $u\in D(A_1)$, the domain of the operator A_1. Denote by D_t^k the kth strong derivative of an $L^{2,r}(\Omega)$-valued vector function of t. Then the solution of the equation (1)': $D_t^2u(-, t)=-A_xu(-, t)+f(-, t)$, with the initial condition $u(-, 0)=Du(-, 0)=0$, may be given formally by the Bochner integral of the $L^{2,r}(\Omega)$-valued vector function: $u(-, t)=\int_{m_0}^t A_1^{-\frac12}\sin((t-\tau)A_1^{\frac12})f(-, \tau)d\tau$. The author proves that this integral admits, for $0\leqq t\leqq T$, a second strong derivative and satisfies (1)' if $A_1^{\frac12}f(-, \tau)$ is Bochner integrable over $(0, T)$ and if $f(-, \tau)$ is strongly continuous in $L^{2,r}(\Omega)$ for almost all τ, $0\leqq\tau\leqq T$. This result is applied to prove the existence of a genuine solution of the original equation (1) with the initial condition $u(-, 0)=\varphi(-)\in L^{2,r}(\Omega)$, $u_t'(-, 0)=\psi(-)\in L^{2,r}(\Omega)$. It is also proved that the method is applicable to the Dirac system of equations $\partial u_i(x, t)/\partial t=(-1)^i(A_xu)_i(x, t)+f_i(x, t)$ $(i=1, 2, \cdots, r)$ in a stationary field.

K. Yosida (Tokyo)

6599:

Krehivs'kiĭ, V. V. Partial differential equations with a principal term. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 10–13. (Ukrainian. Russian and English summaries)

In this note, the author presents the solution of the Cauchy and characteristics problems for an equation of type $A[U]\equiv D^kU(x)+\sum_{s<k}h_s(x)D^sU(x)=f(x)$. The fundamental solution is constructed, and an integral representation of the solution of Cauchy's problem is given.

Author's summary

6600:

Borok, V. M. Equivalent systems of linear partial differential equations with constant coefficients. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 555–558. (Russian)

The system

$$1)\qquad \partial u(x, t)/\partial t=P(i\partial/\partial x)u(x, t),$$

where

$$u(x, t)=\{u_1(x, t), \cdots, u_N(x, t)\},$$

$x=(x_1, \cdots, x_n)$ and $P(i\partial/\partial x)$ is an $N\times N$-matrix whose elements are polynomials in $i\partial/\partial x_1, \cdots, i\partial/\partial x_n$, is said to be equivalent to the system,

$$2)\qquad \partial v(x, t)/\partial t=Q(i\partial/\partial x)v(x, t)$$

if the number N of unknown functions is the same for both systems and if there exists a non-degenerate operator $T(i\partial/\partial x)$, where T is a non-singular $N\times N$ matrix, such that if $u(x, t)$ is a solution of 1), then $v(x, t)=T(i\partial/\partial x)u(x, t)$ is a solution of 2). The author proves that 1) and 2) are equivalent if and only if the matrices P and Q are similar; if 1) and 2) are similar, then every solution $v(x, t)$ of 2) is obtained by operating with a non-degenerate operator $T(i\partial/\partial x)$ on some solution $u(x, t)$ of 1); and finally that systems equivalent to an elliptic system are elliptic.

6601:

Sobolev, S. L. A note on Petrovsky's test for the uniform correctness of the Cauchy problem in the case of partial differential equations. Dokl. Akad. Nauk SSSR **121** (1958), 598–601. (Russian)

Petrovsky's criterion asserts that the equation

$$Lu\equiv\partial^n u/\partial t^n+\sum_{k<n}A_{k,l}(\partial^{k+l}u/\partial t^k\partial x^l)=F$$

with constant coefficients admits a solution which satisfies the conditions $u|_{t=0}=\partial u/\partial t|_{t=0}=\cdots=\partial^{n-1}u/\partial t^{n-1}|_{t=0}=0$ and depends continuously on $A_{k,l}$ and F if and only if, for purely imaginary values of α, all roots of the equation $\lambda^n+\sum_{k<n}A_{k,l}\lambda^k\alpha^l=0$ lie to the left of a certain line $\sigma>\sigma_0$, where $\lambda=\sigma+it$. L is called a Petrovsky operator. The main theorems of the paper establish that every Petrovsky operator has, to within "minor" terms, a canonical representation as a product of Petrovsky operators of a simple type.

R. N. Goss (San Diego, Calif.)

6602:

Sobolev, S. L. On mixed problems for partial differential equations with two independent variables. Dokl. Akad. Nauk SSSR **122** (1958), 555–558. (Russian)

The equation

$$\partial^n u/\partial t^n+\sum A_{k,l}\partial^{k+l}u/\partial t^k\partial x^l=f\quad (k<n,\ l\leqq m)$$

in two independent variables and constant coefficients is assumed to satisfy Petrowski's condition for uniform correctness of the Cauchy problem. The author discusses this problem in the three domains: a) the upper half (x, t) plane, b) the first quadrant and c) the semi-infinite strip $0\leqq x\leqq 1$, $0\leqq t$. The boundary conditions $\partial^i u/\partial t^i=0$ $(i=0, \cdots, n-1)$ at $t=0$ are imposed in all three domains, together with $\sum g_i^{(s)}\partial^i u/\partial x^i=0$ $(i=1, \cdots, m-1, s=1, 2, \cdots, q_-)$ for $x=0$ for the quadrant, and an additional set of q_+ conditions at $x=1$ for the strip. The author derives

conditions under which the problem has a solution and is correctly set. [See also #6601 above; and M. I. Višik and L. A. Lyusternik, Uspehi Mat. Nauk (N.S.) 12 (1957), no. 5(77), 3–122; MR 20 #2539]. The solution rests on application of the Laplace transform to the Green's function. If $\Delta(\lambda, \alpha)=\lambda^n+\sum A_{k,l}\lambda^k\alpha^l$, the author shows the number q_+ and q_- of boundary conditions at $x=0$ and (in case c) at $x=1$ are determined by the number r_+ and r_- of roots of $\Delta(\lambda, \alpha)$ (for any λ) to the right and to the left of the imaginary axis in the α-plane.

A. N. Milgram (Minneapolis, Minn.)

6603:

Lions, Jacques-Louis. Sur certains problèmes mixtes quasi linéaires. I. C. R. Acad. Sci. Paris **246** (1958), 1644–1647.

Let Ω be an open domain of R^n, and $H^k(\Omega)$ the space of functions u whose distribution derivatives D^pu of order $|p|\leq k$ all belong to $L^2(\Omega)$. Thus $H^k(\Omega)$ is normed by $\|u\|_k=(\sum_{|p|\leq k}|D^pu|^2)^{\frac{1}{2}}$, $|u|$ denoting the norm in $L^2(\Omega)$. Let $H_0^k(\Omega)$ be the closure in $H^k(\Omega)$ of the functions with compact supports in Ω, and let V be a closed subspace of $H^m(\Omega)$ containing $H_0^m(\Omega)$. Let, for $0\leq t\leq\mu$, $a_{pq}(x, t)\in L^\infty(\Omega)$ and let the sesquilinear form

$$a(t; u, v)=\sum_{|p|,|q|\leq m}\int_\Omega a_{pq}(x, t)D^quD^pvdx$$

in $u, v \in V$ satisfy the Gårding inequality $a(t; u, v)+\lambda|v|^2 \geq\alpha\|v\|_m^2$. If Ω is a bounded domain with regular boundaries, then the non-linear problem for the operator

$$Lu=\sum_{|p|,|q|\leq m}(-1)^{|p|}D_x^p(a_{pq}(x, t)D_x^qu)+\sum_{|p|\leq m}b_p(x, t, D_x^{m-1}u)D_x^pu+D_tu$$

admits, for $0\leq t\leq\mu$, a weak solution $u(x, t)$ satisfying $\int_0^\mu\|u\|_m^2dt<\infty$, $\int_0^\mu|D_tu|^2dt<\infty$ and $u(x, 0)=0$; i.e., $u(t)=u(x, t)$ satisfies

$$\int_0^\mu a(t; u(t), h(t))dt+\int_0^\mu dt\int_\Omega\sum_{|p|\leq m}b_p(x,t,D_x^{m-1}u(t))D_x^pu(t)\overline{h(t)}dx=\int_0^\mu dt\int_\Omega f(t)\overline{h(t)}dx$$

for a given $L^2(\Omega)$-valued $f(t)$ with $\int_0^\mu|f|^2dt<\infty$ and for every V-valued $h(t)$ with $\int_0^\mu\|h\|_m^2dt<\infty$. The author says that an analogous result may be obtained also in the case where L contains the term D_t^2.

K. Yosida (Tokyo)

6604:

Lions, Jacques-Louis. Sur certains problèmes mixtes quasi linéaires. II. C. R. Acad. Sci. Paris **246** (1958), 1796–1799.

As a sequel to the preceding paper [#6603], a hypothesis (concerning the sesqui-linear form $a(t; u, v)$, the boundaries of the domain Ω and the subspace V defining the boundary condition) is introduced to assure the existence of the weak solution for the operator

$$Lu=\sum_{|p|,|q|\leq m}(-1)^{|p|}D_x^p(a_{pq}(x, t)D_x^qu)+\sum_{|p|\leq m}b_p(x, t, D_x^{2m-1}u)D_x^pu+D_tu.$$

It is stated that analogous results may be obtained for the Schrödinger equation and for systems of equations.

K. Yosida (Tokyo)

6605:

Rachajsky, Borivoj. Théorème de Jacobi pour le système d'équations en involution de Darboux-Lie. Bull. Soc. Math. Phys. Serbie **8** (1956), 7–14. (Serbo-Croatian summary)

Es handelt sich um das Darboux-Liesche Involutionssystem zweiter Ordnung

$$(*)\qquad r+f(x, y, z, p, q, s)=0,\quad t+\varphi(x, y, z, p, q, s)=0.$$

Im Anschluß an bekannte Untersuchungen und Ergebnisse von É. Goursat [Leçons sur l'intégration des équations aux dérivées partielles du second ordre, vol. 2, Hermann, Paris, 1898; p. 73] und N. Saltykow [Glas Srpske Akad. Nauka Od. Prirod.-Mat. Nauka **198** (1950), 37–52; MR **12**, 829] zeigt Verfasser, wie man das allgemeine Integral des zu (*) gehörigen Charakteristikensystems

$$dx=\frac{dy}{f_s}=\frac{dz}{p+f_sq}=\frac{dp}{-f+f_ss}=\frac{dq}{s-\varphi f_s}=\frac{ds}{-D_yf}$$

durch die Gleichungen

$$z=V,\ p=V_x,\ q=V_y,\ s=V_{xy},$$

$$\alpha=D\left(\frac{V, V_x, V_y}{C_i, C_j, C_k}\right):D\left(\frac{V, V_x, V_y}{C_i, C_l, C_k}\right)=C_5\quad(\alpha_x\neq0,\ \alpha_y\neq0)$$

definieren kann. Dabei ist $V=V(x, y, C_1, C_2, C_3, C_4)$ ein vollständiges Integral des Systems (*). Mit D werden die entsprechenden Funktionaldeterminanten bezeichnet.

M. Pinl (Köln)

DIFFERENTIAL ALGEBRA

See 6557, 6558, 6559, 6560.

POTENTIAL THEORY

See also 6537, 6538, 6577.

6606:

✶MacMillan, William Duncan. The theory of the potential. MacMillan's Theoretical Mechanics. Dover Publications, Inc., New York, 1958. xiii+469 pp. $2.25.

An unaltered republication of the first edition [McGraw-Hill, New York, 1930].

6607:

Brelot, Marcel. Le problème de Dirichlet. Axiomatique et frontière de Martin. J. Math. Pures Appl. (9) **35** (1956), 297–335.

Man betrachte einen Greenschen Raum Ω [vgl. Brelot und Choquet, Ann. Inst. Fourier 3 (1951), 199–263; MR **16**, 34] mit einer Metrik, die es erlaubt, Ω durch Hinzufügung des Randes Ω^* zu einem kompakten Raum zu erweitern. Es sei in Ω eine positiv-harmonische Funktion h gegeben. Ist nun f eine auf Ω^* definierte Funktion, so betrachten wir alle in Ω subharmonischen Funktionen u, für welche u/h nach oben beschränkt ist und lim sup $u(P)/h(P)\leq f(Q)$ gilt, sobald $P\in\Omega$ gegen $Q\in\Omega^*$ strebt. Die obere Einhüllende dieser u-Menge sei $\underline{D}_{f,h}(P)$, und es sei $\overline{D}_{f,h}(P)=-\underline{D}_{-f,h}(P)$, wobei $\underline{D}\leq\overline{D}$. Bei Gleichheit, $\underline{D}=\overline{D}=D$, nennt man f h-lösungsfähig (h-résolutive).

Verf. betrachtet im besonderen diejenigen Ω, welche durch folgendes "Axiom" $\mathfrak{A}_h$ ausgezeichnet sind: Jedes auf Ω^* stetige f ist h-lösungsfähig. Er gibt zu diesem Axiom äquivalente Formulierungen, ferner Aussagen über das erweiterte Dirichletproblem sowohl im Falle der Gültigkeit des Axioms wie in allgemeineren Fällen. Gilt $\mathfrak{A}_h$ und ist f h-lösungsfähig, so läßt sich $D_{f,h}(P)$ in der Form $\int_{\Omega^*} f(Q)d\mu_h{}^P(Q)$ darstellen, wobei $d\mu_h{}^P$ als harmonisches h-Maß aufzufassen ist. Eine Randmenge e heißt "h-négligeable" bzw. "faiblement h-négligeable", falls $\overline{D}_{\varphi,h}$ bzw. $\underline{D}_{\varphi,h}$ identisch verschwindet, wobei φ die charakteristische Funktion von e ist. Bei Gültigkeit von $\mathfrak{A}_h$ bedeutet dies, daß das äußere bzw. innere harmonische h-Maß von e verschwindet. Eine funktionentheoretische Anwendung besagt: Sei Ω eine hyperbolische Riemannsche Fläche und daselbst F analytisch, v superharmonisch, sei $h^{-1}\ln(|F|e^{-v})$ nach oben beschränkt und habe den Grenzwert $-\infty$ in einer Randmenge, die nicht h-négligeable ist; dann ist $F\equiv 0$. Im letzten Abschnitt behandelt Verf. seine Theorie in dem Falle, daß eine Martinsche Randtopologie in Betracht kommt. *G. af Hällström* (Zbl **71**, 100)

6608:

Naïm, Linda. Sur le rôle de la frontière de R. S. Martin dans la théorie du potentiel. Ann. Inst. Fourier, Grenoble 7 (1957), 183–281.

The author treats in detail and expands the work announced in previous papers. [C. R. Acad. Sci. Paris **241** (1955), 1907–1910; **242** (1956), 1107–1110, 2695–2698; **243** (1956), 1266–1268; MR **17**, 1073; **18**, 729.] The following is a summary of her principal results. Let G be the Green function of a space Ω, either a connected open subset of a Euclidean space, or, more generally, a Green space in the Brelot-Choquet terminology [Ann. Inst. Fourier. Grenoble **3** (1951), 199–263; MR **16**, 34]. Let y_0 be any point of Ω, and define $K(x, y)=G(x, y)/G(x, y_0)$, $\theta(x, y)=K(x, y)/G(y, y_0)$. The space is completed by adjoining the Martin boundary [Trans. Amer. Math. Soc. **49** (1941), 137–172; MR **2**, 292] to obtain the compact metric space $\hat{\Omega}$, and the function $K(\cdot, y)$ is thereby extended to be continuous on $\hat{\Omega}-\{y_0\}$. The domain of θ is then further extended to allow both x and y to be on the boundary, making θ a positive lower semicontinuous symmetric function with a certain average property.

The function K is used as the kernel in defining the potential of a measure of subsets of Ω (integration with respect to the second argument). The potential is a function on $\hat{\Omega}-\{y_0\}$. A subset E of Ω is said to be thin at a point x_0 of $\hat{\Omega}-\{y_0\}$ if x_0 is not a limit point of E or otherwise if there is a K-potential U with $U(x_0)$ less than the limit inferior of U at x_0 for approach to x_0 on E. In this way Brelot's concept of thinness [Bull. Sci. Math. (2) **68** (1944), 12–36; MR **7**, 15], defined only for points of Ω, is extended to points of the Martin boundary. A boundary point x is minimal, that is, the function $K(x, \cdot)$ is minimal harmonic, if and only if Ω is not thin at x. The function θ is also used as a kernel to define the potential of a measure of subsets of $\hat{\Omega}-\{y_0\}$. The sets containing a point x_0 of $\hat{\Omega}-\{y_0\}$ whose complements are thin at x_0 define the least fine topology making θ potentials continuous. Concepts using this topology will be qualified by "fine". A function on Ω has a fine limit l (also called "pseudo-limit") at a minimal boundary point if and only if it has l as a limit along the complement of a set thin at x_0. If ω is an open subset of Ω, various theorems relating the boundary points of ω to those of Ω are proved. For example, $\Omega-\omega$ is thin at a minimal boundary point of Ω if and only if, for each y in ω, if g is the Green function of ω, $\lim\sup_{x\to x_0} g(x, y)/G(x, y_0)>0$.

Let v be a function on Ω, positive and superharmonic, and let x_0 be a minimal boundary point. Then it is shown (1) that $v/G(\cdot, y_0)$ has the strictly positive number $\lim\inf_{x\to x_0} v/G(\cdot, y_0)$ $(\leqq\infty)$ as fine limit at x_0, and (2) that $v/K(x_0, \cdot)$ has $\inf v/K(x_0, \cdot)$ as fine limit at x_0. These theorems are immediate consequences of the theorems on θ potentials, but are also given direct proofs. If Ω is a half-space, (2) reduces to a theorem of Ahlfors and Heins [Ann. of Math. (2) **50** (1949), 341–346; MR **10**, 522] in the form due to Lelong-Ferrand [Ann. Sci. École Norm. Sup. (3) **66** (1949), 125–159; MR **11**, 176]. If v is the Green potential of a positive measure on Ω, and if h is a strictly positive harmonic function, it is shown that v/h has the fine limit 0 at h almost every (where "h almost every" refers to harmonic measure relative to h) boundary point. It then follows that $G(\cdot, y_0)/h$ has the ordinary limit 0 at h almost every boundary point. This result was proved by Brelot [#6607 above] for the case $h=1$, in a paper which laid the foundations for much of the author's work. If v is positive and superharmonic, and h is strictly positive and harmonic, the reviewer [Bull. Soc. Math. France **85** (1957), 431–458] proved (subsequent to the author's work and using probability methods) that v/h has a fine limit at h almost every boundary point, obtaining also an extended theorem in which h is allowed to be superharmonic. For such theorems in a more general context, see another paper by the reviewer [Illinois J. Math. **2** (1958), 19–36].

The author then studies the Dirichlet problem for functions which are quotients of harmonic functions divided by a preassigned positive harmonic function h, concentrating on approach to the Martin boundary in terms of the fine topology rather than in terms of the Martin topology. (The reviewer's theorem in the first reference above that the Dirichlet solution has the prescribed boundary function as a fine limit at h almost every boundary point clarifies some of this work.) Finally, the Dirichlet problem is treated using any boundary Γ which together with Ω forms a compact metric space, and it is shown that a correspondence between points of Γ and of the Martin boundary can be constructed such that every Dirichlet problem for a boundary function on Γ can be reduced to one on the Martin boundary with a corresponding boundary function. One side result is that, if Ω is a set in a Euclidean space, bounded if the dimensionality is more than 2, the minimal harmonic functions are unbounded. *J. L. Doob* (Urbana, Ill.)

6609:

Hervé, Rose-Marie. Sur le problème de Dirichlet dans un espace de Green. C. R. Acad. Sci. Paris **247** (1958), 401–404.

Si un espace de Green Ω est partout dense dans un espace métrique compact $\hat{\Omega}$, on introduit la frontière $\Gamma=\hat{\Omega}-\Omega$ et une fonction harmonique $h>0$ dans Ω, pour laquelle on suppose satisfait l'axiome appelé $\mathfrak{A}_h$, qui permet le développement du problème de Dirichlet (relativisé pour h) [voir Brelot, #6607 ci-dessus]. L'A. étudie la mesure harmonique correspondante $\mu_h{}^x$ en introduisant la densité $\rho_h{}^x(y)$ de $\mu_h{}^x$ relativement à $\mu_h{}^{x_0}$ au point $y\in\Gamma$. On peut faire en sorte que $\rho_h{}^x(y)$ soit définie p.p. $d\mu_h{}^{x_0}$ indépendamment de x et harmonique en x et on peut l'exprimer au moyen d'une intégrale faisant intervenir la frontière de Martin. Cette densité est

utilisée pour caractériser la "h-action" d'un ensemble $\alpha C\Gamma$ sur un filtre de Ω (en particulier celui des voisinages d'un point-frontière). Cela complète une étude "d'action à distance" dans le problème de Dirichlet, qui, dans les conditions générales précédentes, a été développée par L. Naïm [#6608 ci-dessus]. *M. Brelot* (Paris)

6610:

Brelot, M. Sur l'allure à la frontière des fonctions sousharmoniques ou holomorphes. Ann. Acad. Sci. Fenn. Ser. A. I, no. 250/4 (1958), 9 pp.

Cet article prolonge un travail antérieur de l'auteur [voir #6607 ci-dessus]. Il est caractérisé par l'utilisation des notions d'effilement et de pseudo-limite en un point de la frontière de Martin d'un espace de Green [L. Naïm, #6608 ci-dessus]. Cela permet d'énoncer une forme améliorée du principe du maximum, qu'on applique à l'étude du comportement à la frontière des fonctions sousharmoniques dont le quotient par une fonction harmonique positive est borné supérieurement. Une autre application concerne la convergence de suites de fonctions holomorphes sur une surface de Riemann hyperbolique.

J. Deny (Strasbourg)

6611:

Forte, Bruno. Su una particolare classe di funzioni spaziali armoniche ortonormali nel cerchio $z=0$, $x^2+y^2\leqq a^2$. Ann. Scuola Norm. Sup. Pisa (3) 11 (1957), 265–277.

Si considerino nello spazio (x, y, z) i potenziali newtoniani

$$V_s(x, y, z)=\int_\sigma \mu^s \frac{1}{r}\, d\sigma_{\xi\mu} \quad (s=-1, 1, 3, 5, \cdots),$$

dove $r^2=(x-\xi)^2+(y-\eta)^2+z^2$, σ è il cerchio $x^2+y^2\leqq a^2$ e $\mu(\xi, \eta)=(1+(\xi^2+\eta^2)/a^2)^{\frac{1}{2}}$.

Vengono stabilite in modo assai rapido e semplice formule ricorrenti per la determinazione dei coefficienti delle combinazioni lineari dei potenziali V_s sopradetti, le cui restrizioni sul cerchio σ costituiscono un sistema ortonormale su σ. Il risultato può servire per la risoluzione di certi problemi al contorno armonici nello spazio e nel semispazio. *E. Magenes* (Genoa)

6612:

Paria, Gunadhar. Notes on a mixed problem in potential theory. Bull. Calcutta Math. Soc. 49 (1957), 95–97.

Détermination de la fonction harmonique dans le domaine plan $|z|>1$, $y>0$, nulle à l'infini, prenant des valeurs données sur le demi-cercle $|z|=1$, $y\geqq 0$, et ayant une dérivée normale nulle sur le reste de la frontière.

J. Deny (Strasbourg)

6613:

Pachale, Helmut. Einige Hilfsbetrachtungen zu einem biharmonischen Randwertproblem. Math. Nachr. 18 (1958), 218–221.

Il lavoro contiene solo due lemmi assai particolari su alcune proprietà di regolarità di funzioni definite sulla frontiera di un dominio dello spazio ordinario, lemmi che serviranno all' Autore in un successivo lavoro dedicato a un problema al contorno biarmonico.

E. Magenes (Genoa)

6614:

Bramble, James H. Continuation of biharmonic functions across circular arcs. J. Math. Mech. 7 (1958), 905–924.

Explicit formulae are derived for the continuation of a biharmonic function w across a circular arc Q when the following conditions are satisfied on Q: (A) $w=M(w)=0$, (B) $M(w)=V(w)=0$, (C) $w_r=V(w)=0$, where

$$M(w)=\Delta w+(1-\sigma)\sigma^{-1}w_{rr},$$
$$V(w)=(\Delta w)_r+(1-\sigma)a^{-2}[w_{\theta\theta r}-a^{-1}w_{\theta\theta}],$$

Δ being the Laplace operator and a the radius of the circle. — Application (σ=Poisson's ratio): Elastic plate with (A) simply supported, (B) free, (C) sliding clamped edge Q. *A. Huber* (Munchenstein)

FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS

See 6488.

SEQUENCES, SERIES, SUMMABILITY

See also 6662.

6615:

Mordell, L. J. On the evaluation of some multiple series. J. London Math. Soc. 33 (1958), 368–371.

Denote by S_r for $r>1$ the series

$$S_r=\frac{1}{1^r}+\frac{1}{2^r}+\cdots+\frac{1}{n^r}+\cdots.$$

The author gives some new results concerning the problem of expressing various series in terms of S_r. Set, for a given positive integer r and any $a>-r$,

$$T(a)=\sum_l 1/l_1 l_2\cdots l_r(l_1+l_2+\cdots+l_r+a),$$

where the summation is taken over all sets $(l_1, l_2, \cdots, l_r)$ of integers l_i satisfying $1\leqq l_i<\infty$ $(i=1, 2, \cdots, r)$. Then the following theorem is representative of those proven by the author.

$$T(a)=r!\left(1-\frac{a-1}{1!2^{r+1}}+\frac{(a-1)(a-2)}{2!3^{r+1}}-\cdots\right),$$

where the series converges absolutely.

V. F. Cowling (Lexington, Ky.)

6616:

Robertson, A. P. On rearrangements of infinite series. Proc. Glasgow Math. Assoc. 3 (1958), 182–193.

Analog den klassischen Umordnungsproblemen werden für ein beliebiges lineares Limitierungsverfahren $A=(a_{ik})$ die folgenden Probleme behandelt: (1) Welche Bedingungen muss eine konvergente Reihe erfüllen, damit jede Umordnung A-summierbar ist? (2) Wie muss eine spezielle Umordnung beschaffen sein, damit sie jede konvergente Reihe in ein A-summierbare Reihe umordnet? Es wird gezeigt, dass für (1) sich dieselbe Bedingung ergibt wie im klassischen Fall, nämlich die absolute Konvergenz der Reihe. Zu (2) werden notwendige und hinreichende Bedingungen angegeben, die deswegen etwas schwierig zu formulieren sind, weil in die gesuchten Bedingungen über die Umordnung natürlich das betrachtete Verfahren in komplizierter Weise eingeht. Jedoch werden in Spezialfällen, wenn nämlich das Verfahren A positiv ist ($a_{ik}\geqq 0$) und eine "schnelle" Umordnung vorliegt, für Anwendungen geeignete Kriterien gegeben.

K. Endl (Giessen)

6617:

Lorch, Lee. Supplement to a theorem of Cesàro. Scripta Math. 23 (1957), 163–165 (1958).

The author considers two conditions given in G. H. Hardy's *Divergent series* [Clarendon Press, Oxford, 1949; MR 11, 25] which ensure that when the $(\bar{N}, p_n)$ limit of a sequence s_n, defined by

$$\lim_{m\to\infty} \frac{p_0 s_0 + p_1 s_1 + \cdots + p_m s_m}{p_0 + p_1 + \cdots + p_m},$$

exists and is finite, then the corresponding $(\bar{N}, q_n)$ exists and is equal. He shows that one condition can be extended to the case of infinite limits, but not the other. This question arose in connection with work on the transfinite diameter. *H. G. Eggleston* (Pinner Middlesex)

APPROXIMATIONS AND EXPANSIONS

See also 6772, 6773.

6618:

Berman, D. L. A method of constructing interpolation formulas. Dokl. Akad. Nauk SSSR 124 (1959), 11–14. (Russian)

Let $\omega_k(x)$ $(k=0, 1, \cdots)$ be a system of polynomials which are orthonormal on $[-1, 1]$ with respect to the weight $p(x) \geqq 0$. $\lambda_k^{(n)}$ are given constants. Set

$$R_{n,n}(x, t) = \sum_{k=0}^{n} \lambda_k^{(n)} \omega_k(x) \omega_k(t),$$

$$\sigma(f; x) = \int_{-1}^{1} f(t) R_{n,n}(x, t) p(t) dt.$$

Let this integral be approximated by a Gauss-Jacobi quadrature rule of order s with abscissas $x_k^{(s)}$ $(k=1, 2, \cdots, s)$ and weights $\rho_k^{(s)}$:

$$\bar{\sigma}_n(f; x) = \sum_{k=1}^{s} \rho_k^{(s)} f(x_k^{(s)}) R_{n,n}(x, x_k^{(s)}).$$

Theorem: If $p(x)$ satisfies $0 < A \leqq p(x)\sqrt{(1-x^2)} \leqq B$, $-1 \leqq x \leqq 1$, if for any f which is continuous on $[-1, 1]$ we have $\sigma_n(f; x) \to f(x)$ uniformly, and if $2s-n \to \infty$ $(n\to\infty)$, then $\bar{\sigma}_n(f; x) \to f(x)$ uniformly. *P. Davis* (Washington, D.C.)

6619:

Chandrasekhar, S. On the expansion of functions satisfying four boundary conditions. Mathematika 4 (1957), 140–145.

The author exhibits a biorthogonal set of functions suitable for expanding a given $F(r)$ which is to satisfy the four conditions $F(r)=0$ at $r=1$ and $r=\eta<1$, $F'(r)=0$ at $r=1$ and $r=\eta$. Specifically, $F(r)=\sum_{j=1}^{\infty} A_j f_j(r)$, where

$$f_j(r) = \alpha_j^{-1} \mathscr{C}_\nu(\alpha_j r) + (2\nu)^{-1}(\beta_j r^\nu + \gamma_j / r^\nu) \quad (j=1, 2, \cdots),$$

$$\mathscr{C}_\nu(\alpha_j r) = J_{-\nu}(\alpha_j \eta) J_\nu(\alpha_j r) - J_\nu(\alpha_j \eta) J_{-\nu}(\alpha_j r),$$

$$A_j = \alpha_j N^{-1} \int_\eta^1 F(r) \mathscr{C}_\nu(\alpha_j r) dr,$$

$$\beta_j = -\mathscr{C}_{\nu-1}(\alpha_j), \ \gamma_j = -\mathscr{C}_{\nu+1}(\alpha_j),$$

and the sequence $\{\alpha_j\}$ are the (real) roots of a determinantal equation in α involving the Bessel functions $J_{\pm(\nu-1)}(r)$ at $r=\alpha$ and $r=\alpha\eta$. A similar, and simpler, expansion $F(x)=\sum_{j=1}^{\infty} A_j f_j(x)$ is given, where

$$A_j = 8 \int_0^1 F(x) X_j(x) dx,$$

$$f_j(x) = \alpha_j^{-1} \sin \alpha_j x - \tfrac{1}{2} \cos \alpha_j x - x + \tfrac{1}{2} \quad (j=1, 2, \cdots)$$

$$X_j(x) = \alpha_j^{-1} \sin \alpha_j x - \tfrac{1}{2} \cos \alpha_j x,$$

and $\{\alpha_j\}$ are the (real) roots of $\tan \tfrac{1}{2}\alpha = \tfrac{1}{2}\alpha$. This is appropriate if $F(x)$ is to satisfy $F(0)=F(1)=F'(0)=F'(1)=0$. *A. B. Novikoff* (Menlo Park, Calif.)

6620:

Mamedov, R. G. Weighted approximation in the space $L_p(-\infty, \infty)$. Trudy Azerbaĭdžan. Gos. Ped. Inst. Lenin. 2 (1955), 154–158. (Russian)

The author obtains a sufficient condition for $\varphi(x) > 0$, $-\infty < x < +\infty$, in order that polynomials be dense in the space L^p on $(-\infty, +\infty)$ with the weight $1/\varphi(x)$. His condition is very close to the necessary and sufficient condition given by Pollard [Proc. Amer. Math. Soc. 6 (1955), 402–411; MR 16, 1104]. *G. G. Lorentz* (Syracuse, N.Y.)

6621a:

Džafarov, A. S. Best approximation in the mean to functions of several variables by entire functions of finite degree. Trudy Azerbaĭdžan. Gos. Ped. Inst. Lenin. 2 (1955), 110–116. (Russian)

6621b:

Džafarov, A. S. Mean-square best approximation of periodic functions of several variables by trigonometric polynomials. Trudy Azerbaĭdžan. Gos. Ped. Inst. Lenin. 2 (1955), 159–162. (Russian)

In the first paper, the author treats by methods of Nikol'skiĭ [Trudy Mat. Inst. Steklov., v. 38, pp. 244–278, Izdat. Akad. Nauk SSSR, Moscow 1951; MR 14, 32] the approximation of functions $f(x, y)$ of classes ML^p (and of more general classes) by means of entire functions. Here $f \in ML^p$ if

$$\|f\| = \sup \operatorname{ess}_x \left\{ \int_{-\infty}^{+\infty} |f(x, y)|^p dy \right\}^{1/p} < +\infty.$$

In the second paper, the L^2 approximation by trigonometric polynomials of a termwise integrated Fourier series is treated, when the original series has a given L^2 norm. *G. G. Lorentz* (Syracuse, N.Y.)

6622:

Voronovskaya, E. V. On Čebyšev's approximation of analytic functions by algebraic polynomials. Dokl. Akad. Nauk SSSR 121 (1958), 206–209. (Russian)

Using notions developed previously [Voronovskaya, same Dokl. 114 (1957), 927–929; MR 20 #1151], the author describes a method to find the polynomial of best approximation of a given degree on [0, 1] to a function $f(x)$ analytic for $|x| \leqq 1$. *G. G. Lorentz* (Syracuse, N.Y.)

6623:

Zuhovickiĭ, S. I.; and Èskin, G. I. Chebyshev approximation in a commutative Hilbert ring. Dokl. Akad. Nauk SSSR 119 (1958), 1074–1076. (Russian)

Let Q be a compact Hausdorff space, H a commutative Hilbert ring ($=H^*$ algebra), $\varphi_k(q)$, $f(q)$ $(k=1, \cdots, n)$ continuous functions on Q with values in H. Polynomials $P(q)=\sum_{k=1}^{n} a_k \varphi_k(q)$ are considered. Let T be the subspace of H^n which consists of complexes $(a_1, \cdots, a_n)$, $a_k \in H$ with $\sum a_k \varphi_k(q) \equiv 0$, S the orthogonal complement of T in H^n. Let F_φ be the set of continuous f which possess a polynomial of best approximation minimizing $\max_q \|f(q)-P(q)\|$. Following results are stated. (a) All continuous functions are in F_φ if and only if S is finite-dimensional. (b) In this case, let e_α be a basis of orthogonal irreducible hermitean idempotents in H, $\varphi_k(q)=\sum \varphi_{k_\alpha}(q) e_\alpha$, where $\varphi_{k_\alpha}(q)$ are complex valued continuous functions. If l is the number of φ_{k_α} not identically zero, and dim $S=tl$

is a multiple of l, each continuous f has a unique polynomial of best approximation if and only if each P, not identically zero, vanishes in at most t points of Q. Also in case $S=H^n$ there are similar theorems. Results of the text parallel those of Zuhovickiĭ and Stečkin [same Dokl. **106** (1956), 385–388; MR **18**, 222], where H was an arbitrary Hilbert space and polynomials had scalar coefficients. *G. G. Lorentz* (Syracuse, N.Y.)

6624:

Eskin, G. I.; and Zuhovic'kiĭ, S. I. Some theorems on the Tchebycheff approximation of functions with values belonging to a commutative C^*-algebra. Dopovidi Akad. Nauk Ukraïn. RSR **1958**, 368–371. (Ukrainian. Russian and English summaries)

Let $\varphi_k(q)$, $f(q)$ $(k=1, \cdots, n)$ be continuous functions on a compact Hausdorff space with values in a completely regular Banach ring R (=Banach algebra with involution satisfying $\|xx^*\|=\|x\|^2$) which has a unity. In terms of distributions of spectral values of appropriate elements of R, necessary conditions are given in order that (a) $\sum a_k\varphi_k(q)$ be a polynomial of best approximation to $f(q)$, and (b) each f has a unique polynomial of best approximation. The results follow at once by combining approximation theorems of Kolmogoroff [see Zuhovickiĭ and Kreĭn, Uspehi Mat. Nauk (N.S.) **5** (1950), no. 1(35), 217–229; MR **11**, 662] with the Gelfand-Naĭmark representation theorem for rings R.

G. G. Lorentz (Syracuse, N.Y.)

6625:

Sieklucki, K. Topological properties of sets admitting the Tschebycheff systems. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 603–606.

Let $C(Z)$ be the space of all real continuous functions on the compact Hausdorff space Z. A linearly independent set $\{x_i\}_{i=1}^n \subset C(Z)$ is called a Čebyšev system of order n if the function $\sum_{i=1}^n c_ix_i$ has at most $n-1$ zeros in Z whenever $\sum_{i=1}^n |c_i|>0$. The author gives a proof that if $C(Z)$ contains a Čebyšev system of order n $(n>1)$, then Z is homeomorphic with a subset of the circumference of a circle. If this subset is the entire circumference, then n is odd. This generalizes a result of J. C. Mairhuber [Proc. Amer. Math. Soc. **7** (1956), 609–615; MR **18**, 125], who proved the same statement under the assumption that Z was a subset of E^k. {The author does not refer to Mairhuber's paper.} The theorem has also been proved independently by P. C. Curtis [Pacific J. Math. **9** (1959), 1013–1027].

R. R. Phelps (Princeton, N.J.)

6626:

Pták, Vlastimil. A remark on approximation of continuous functions. Czechoslovak Math. J. **8(83)** (1958), 251–256. (Russian summary)

Let E be an n-dimensional subspace of $C(T)$, T compact Hausdorff. The classical theorem of Haar states that, for $T=[0, 1]$, each x in $C(T)$ has a unique nearest point in E if and only if each y in E, $y\neq0$, has at most $n-1$ zeros in T. The author gives a short proof of this result for the case when T is compact Hausdorff, using elementary theorems on finite dimensional convex sets. The theorems of Čebyšev and de la Vallée-Poussin concerning the function in E nearest to x (for the case $T=[0, 1]$) follow easily.

R. R. Phelps (Princeton, N.J.)

6627:

***Rosser, J. Barkley. Some sufficient conditions for the existence of an asymptotic formula or an asymptotic expansion.** On numerical approximation. Proceedings of a Symposium, Madison, April 21–23, 1958, pp. 371–387. Edited by R. E. Langer. Publication no. 1 of the Mathematics Research Center, U. S. Army, the University of Wisconsin. The University of Wisconsin Press, Madison, 1959. x+462 pp. (1 insert) $4.50.

The problem of deriving the asymptotic behavior, for large z, of $\int_a^b f(x)\exp(zg(x))dx$ is considered. Some classical methods are discussed briefly. A generalization of Watson's lemma is given for the case where x and z are complex. For the case where x is real, certain weak conditions are imposed on f and g, and a general theorem is derived. The latter is adapted to the case where f and g depend weakly on z, and numerical results are given to show how remarkably good the asymptotic approximations can be in such cases. *T. E. Hull* (Pasadena, Calif.)

FOURIER ANALYSIS

6628:

Ibragimov, I. I. Extremum problems in the class of trigonometric polynomials. Dokl. Akad. Nauk SSSR **121** (1958), 415–417. (Russian)

Inequalities for trigonometric polynomials T_n in k variables, which for simplicity we formulate for $k=1$. For $1\leq p\leq 2$,

$$\left|\int_0^{2\pi} T_nKdt\right|\leq\|T_n\|_p\Big(\sum_{-n}^{n} |b_i|^{p'}\Big)^{1/p'},$$

where b_i are the Fourier coefficients of K, and

$$\|T_n\|_{p'}\leq[(2n+1)/2\pi]^{1/p-1/p'}\|T_n\|_p,\ p\leq p'.$$

{No proofs are given, but both results follow easily by combining the Hölder and the Riesz-Hausdorff inequalities}. *G. G. Lorentz* (Syracuse, N.Y.)

6629:

Malyavko, K. F. Convergence of Fourier series in systems of type $\{\varphi(nx)\}$ close to a trigonometric system. Dokl. Akad. Nauk SSSR (N.S.) **118** (1958), 29–32. (Russian)

Let $L_2'[-\pi, \pi]$ be the space of odd, complex-valued functions $\phi(x)$ square-summable on $[-\pi, \pi]$. Let

$$\phi(x)\sim \sum_{k=1}^{\infty} b_k \sin kx$$

be the Fourier sine series for $\phi(x)$. If $b_1\neq0$, construct the conjugate system $\{\phi(nx)\}$ defined by $(\phi(nx), g_k(x))=\delta_{nk}$, where δ_{nk} is the Kronecker delta.

Now, for $F(x)$ in $L_2'[-\pi, \pi]$, consider the Fourier series

$$F(x)\sim \sum_{k=1}^{\infty} a_k\phi(kx),$$

with $a_k=\int_{-\pi}^{\pi} F(x)g_k(x)dx$.

The purpose of the present note is to find conditions on $F(x)$ and on $\{\phi(nx)\}$ to ensure uniform convergence everywhere, almost everywhere, or in the mean, to $F(x)$.

Six theorems are given. Theorem 1 states that convergence in the mean holds for every $F(x)$ in L_2'. Theorem 2 is: for uniform convergence to $F(x)$ it is necessary and sufficient

that the corresponding trigonometric series $\sum_{k=1}^{\infty} a_k \sin kx$ converge uniformly. Theorems 3 and 4 relate other properties of $\{\phi(nx)\}$ to the same trigonometric series. Theorem 5 is: if $\sum_{k=2}^{\infty} a_k^2 \ln k$ converges, then the above series for $F(x)$ converges almost everywhere. The final theorem is: for every odd continuous $F(x)$ the above Fourier series in $\{\phi(nx)\}$ is $(C, 1)$ summable uniformly to $F(x)$.

6630:

Stein, Elias M. A maximal function with applications to Fourier series. Ann. of Math. (2) 68 (1958), 584–603.

Let f be Lebesgue integrable on $[0, 2\pi]$ and let $U(\rho, \theta)$ be its Poisson integral. The maximal functions that are studied are defined by

$$M_\lambda(f, \theta) = \left[\sup_{\rho<1} (1-\rho)^{\lambda-1} \int_{\pi \geq |t| \geq 1-\rho} |U(\rho, \theta+t)|^2 dt/|t|^\lambda\right]^{\frac{1}{2}},$$

where $1<\lambda\leq 2$. The fundamental inequalities that are obtained are as follows for $f \in L_p$, $1\leq p<2$ and $\lambda=2/p$.

(i) $$\int_0^{2\pi} (M_\lambda(f, \theta))^q d\theta \leq A(\lambda, q) q(q-p)^{-1} \int_0^{2\pi} |f(\theta)|^q d\theta,$$

if $p<q<\infty$.

(ii) $$\int_0^{2\pi} (M_\lambda(f, \theta))^p d\theta \leq A(\lambda) \int_0^{2\pi} |f(\theta)|^p \log^+ |f(\theta)| d\theta + A'(\lambda),$$

(iii) $$\left(\int_0^{2\pi} (M_\lambda(f, \theta))^q d\theta\right)^{1/q} \leq A(p, q) \left(\int_0^{2\pi} |f(\theta)|^p d\theta\right)^{1/p}$$

if $0<q<p$.

One application of these maximal functions is the following. Let σ_n^α be the Cesàro mean of order α of the Fourier series for the function $\Phi \in H_p$, $0\leq p<1$. Let $\sigma_*^\alpha(\theta) = \sup |\sigma_n^\alpha(\theta)|$ for $n\geq 0$ and let $\alpha = (1/p)-1$. Then

(iv) $$\int_0^{2\pi} (\sigma_*^\alpha(\theta))^p d\theta \leq A(p) \int_0^{2\pi} |\Phi(e^{i\theta})|^p \log^+ |\Phi(e^{i\theta})| d\theta + B(p)$$

if $\Phi \in H^p \log^+ H$, and

(v) $$\left(\int_0^{2\pi} (\sigma_*^\alpha(\theta))^q d\theta\right)^{1/q} \leq A(p) \left(\int_0^{2\pi} |\Phi(e^{i\theta})|^p\right)^{1/p}$$

if $0<q<p$.

Formula (iv) was conjectured by Zygmund [Proc. London Math. Soc. 47 (1942), 326–350; MR 4, 76]. For $0<p\leq\frac{1}{2}$ the conjecture had been established independently by G. Sunouchi [Tôhoku Math. J. 7 (1955), 96–109; MR 17, 361] and by Zygmund [Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 208–212; MR 17, 1080].

Applications of the maximal theorem are also made to strong summability of Fourier series.

P. Civin (Eugene, Ore.)

6631:

Timan, M. F. Inverse theorems of the constructive theory of functions in L_p spaces ($1\leq p\leq\infty$). Mat. Sb. N. S. 46(88) (1958), 125–132. (Russian)

For a 2π-periodic function $f \in L^p$ $(1\leq p<+\infty)$, let $E_n(f)$ be its degree of approximation in the L^p-norm by trigonometric polynomials. In terms of $E_n(f)$, estimates of the smoothness modulus

$$\omega_k(f; t) = \sup_{|h|\leq t} \left\{\int_0^{2\pi} |\Delta^k f(x, h)|^p dx\right\}^{1/p}$$

are given:

(1) $$\omega_k(f; n^{-1}) \leq M_{pk} n^{-k} \varepsilon_n;$$

here

$$\varepsilon_n = \varepsilon_{np} = \left\{\sum_{\nu=1}^{n} \nu^{kp-1} E_{\nu-1}(f)^p\right\}^{1/p}$$

for $1<p\leq 2$, and $\varepsilon_n = \varepsilon_{n2}$ for $p\geq 2$;

(2) $$\omega_k(f^{(r)}; n^{-1}) \leq M_{pkr} \left\{\delta_n + \sum_{\nu=n+1}^{\infty} \nu^{r-1} E_{\nu-1}(f)\right\};$$

here

$$\delta_n = \delta_{np} = n^{-k} \left\{\sum_{\nu=1}^{n} \nu^{p(k+r)-1} E_{\nu-1}(f)^p\right\}^{1/p}$$

for $1<p\leq 2$, and $\delta_n = \delta_{n2}$ for $2\leq p$. There is also a discussion of the relations between $\omega_k(f^{(r)}; h)$ and $\omega_{k+1}(f; t)$.

G. G. Lorentz (Syracuse, N.Y.)

6632:

Telyakovskiĭ, S. A. Approximation of differentiable functions by de la Vallée Poussin's sums. Dokl. Akad. Nauk SSSR 121 (1958), 426–429. (Russian)

De la Vallée-Poussin sums are

$$v_{n,m}(f, x) = m^{-1} \sum_{k=n-m}^{n-1} s_k(f, x),$$

where s_k is the kth partial sum of the Fourier series of f. Let

$$V_{nm}(\mathfrak{M}) = \sup_{f \in \mathfrak{M}} \|f(x) - v_{nm}(f, x)\|$$

with uniform norm. The asymptotic behavior of $V_{nm}(\mathfrak{M})$ for $m=n$ or $m=1$ is known if $\mathfrak{M}$ is the class W^r of functions $f(x)$ with absolutely continuous derivative $f^{(r-1)}(x)$, such that $|f^{(r)}(x)|\leq 1$ almost everywhere, or if $\mathfrak{M}$ is the conjugate class $\overline{W}^r$. Asymptotic expressions given here for $V_{nm}(W^r)$ and $V_{nm}(\overline{W}^r)$ under the assumption $m/n\to\theta$, $0<\theta<1$ are of the form $c(r, \theta)n^{-r}+o(n^{-r})$. Also the cases $\theta=1$ with $n-m=$const or $n-m\to\infty$ are discussed. Integral representations such as

$$V_{nm}(\overline{W}^r) = \frac{2}{\pi m} \int_0^\infty \left| \int_{u_r}^\infty \cdots \int_{u_1}^\infty u^{-2} [\sin nu - \sin(n-m)u] du \cdots du_{r-1} \right| du_r + O(m^{-r}(n-m)^{-r})$$

are the kernel of the proof.

G. G. Lorentz (Syracuse, N.Y.)

6633:

Wintner, Aurel. On the sine approximations to convex arches. Scripta Math. 23 (1957), 153–156 (1958).

The convex arches of the title are positive functions, concave downward over $(0, \pi)$, with

$$f(0+) = f(\pi-) = 0, \quad \pi^{-1} \int_0^\pi f^2(x) dx = 1,$$

and f symmetric about $\frac{1}{2}\pi$. The author asks for the largest possible value of the mean-square error when f is approximated by the best sine curve $y=b \sin x$. He finds that the error is largest when $f(x)$ is the roof function which is $12^{\frac{1}{2}} x/\pi$ on $(0, \pi/2)$ (continued by reflection).

R. P. Boas, Jr. (Evanston, Ill.)

6634:

✱Wiener, Norbert. The Fourier integral and certain of its applications. Dover Publications, Inc., New York, 1959. xi+201 pp. $1.50.

An unaltered republication of the 1933 edition [University Press, Cambridge].

6635:

Kovan'ko, A. S. Application of the Riesz-Fischer theorem to the almost-periodic functions of Weyl. L'viv. Derž. Univ. Dopovidi ta Povidomlennya **1955**, no. 5, 93. (Russian)

If the sequences $\{a_n\}$ and $\{\lambda_n\}$ (a_n complex, λ_n real) are such that $\sum_{n=1}^{\infty} |a_n|^2 = A < \infty$ and

$$\sum_{\substack{k,l=1\\ k\neq l}}^{\infty} \frac{a_k \cdot \bar{a}_l}{\lambda_k - \lambda_l} (e^{i(\lambda_k-\lambda_l)(a+T)} - e^{i(\lambda_k-\lambda_l)})^T$$

is bounded in absolute value, then the Riesz-Fischer theorem holds for the space of W_2-almost periodic functions.

6636:

Edwards, R. E. Bounded functions and Fourier transforms. Proc. Amer. Math. Soc. **9** (1958), 440–446; erratum, 1000.

Let X be a compact abelian group, Y its dual group and S a subset of Y. When does every bounded function on S coincide with the Fourier transform of some bounded measure on X? The author gives the following criterion: Theorem: It suffices that the characteristic function of each subset of S is equal on S to the transform of some Radon measure on X. The following lemma is used in the proof. Let m be a positive Radon measure on Y. If a function h, locally integrable for m, satisfies $\int |h\hat{f}| dm < \infty$ for all f in $L^1(X)$, then h is integrable for m and there exists a constant c such that

$$\int |h| dm \leq c \cdot \sup_{\|f\|_1 \leq 1} \int |h\hat{f}| dm$$

for all such h. *J. Wermer* (Cambridge, Mass.)

INTEGRAL TRANSFORMS

See also 6545.

6637:

Evans, G. C. Calculation of moments for a Cantor-Vitali function. Amer. Math. Monthly **64** (1957), no. 8, part II, 22–27.

The author finds moments of the forms

$$M_n = \int_0^1 x^n p(x) dx \text{ and } m_n = \int_0^1 x^n dp(x)$$

for functions $p(x)$ which have certain properties of symmetry. In particular, the author considers the function $p(x)$ defined on the Cantor "middle third set" as follows: $\frac{1}{2}$ on $\frac{1}{3}<x<\frac{2}{3}$; $\frac{1}{4}$ on $1/9<x<2/9$; $\frac{3}{4}$ on $7/9<x<8/9$; $1/8$ on $1/27<x<2/27$; $3/8$ on $7/27<x<8/27$, $\cdots$. For this "middle third" function $p(x)$, the author derives the formula

$$2M_n = \frac{1}{n+1} + \frac{1}{3^{n+1}-1} \left\{(2+M)^n - M_n\right\},$$

expressing in symbolic form a binomial development in which M^k stands for M_k, and M^0 for M_0 instead of 1. The author also derives moment formulas for a slight generalization of the function $p(x)$.

M. Dresher (Pacific Palisades, Calif.)

6638:

Reid, Walter P. Finite transforms. SIAM Rev. **1** (1959), 44–46.

The author develops heuristically, and illustrates by an example, a generalization of finite Fourier transformations to finite transformations generated by the eigenfunctions of certain non-singular self-adjoint differential operators. *P. G. Rooney* (Toronto, Ont.)

INTEGRAL AND INTEGRODIFFERENTIAL EQUATIONS

6639:

Pokornyĭ, V. V. On the construction of a branching equation. Voronež. Gos. Univ. Trudy Sem. Funkcional. Anal. **1957**, no. 5, 15–21. (Russian)

The author derives the system of bifurcation equations (branching equations or Verzweigungsgleichungen) for a nonlinear integral equation the kernel of which is analytic in the unknown function and a scalar parameter. He indicates how the solutions of the bifurcation equations can be obtained by use of the Weierstrass preparation theorem and the Puiseux expansion. The author is evidently unaware that this method of solution was introduced and developed in more detail by Lefschetz [Comment. Math. Helv. **28** (1954), 341–345; *Differential equations: geometric theory*, Interscience, New York-London, 1957; pp. 163–169; MR **16**, 822; **20** #1005].

J. Cronin (New York, N.Y.)

6640:

Pogorzelski, W. Sur les équations intégrales résolubles sans limitation et leurs applications aux équations différentielles. J. Math. Pures Appl. (9) **37** (1958), 21–40.

The idea of the paper is to formulate an analytical condition leading to satisfaction of the hypotheses of the Schauder fixed point theorem. Thus, let $F(x, y, u)$ be continuous on $A \times A \times R$, where A is a finite segment of the axis of reals, R. The key restriction is

$$(1) \qquad |F(x, y, u)| < K(|u|),$$

where $K(\rho)/\rho \to 0$ for $\rho \to \infty$. Then

$$\phi(x) = T\phi = \int_A F(x, y, \phi(y))|x-y|^{-\beta} dy$$

admits at least one solution in $C(A)$ independently of the length of A. Let S_a be the closed ball $\{u| \, \|u\| \leq a\}$ in the C norm. Then $\psi(x) = (T\phi)(x)$ is a continuous transformation on $C(A)$ to $C(A)$. For a sufficiently large, in view of (1), $T(S_a) \subset S_a$. Moreover, since the modulus of continuity of ψ depends only on the bounds of $|F|$, equicontinuity is ensured and, therefore, TS_a is compact. The Schauder theorem guarantees a fixed point. (The author's analysis is actually for A bounded and measureable in R^n and $\phi = (\varphi_1, \cdots, \varphi_n)$ on A to R^n.) Several partial differential equation applications are given.

D. G. Bourgin (Urbana, Ill.)

6641:

Busbridge, I. W. On the H-functions of S. Chandrasekhar. Quart. J. Math. Oxford Ser. (2) **8** (1957), 133–140.

This function is the solution of an integral equation of the type

$$\frac{1}{H(\mu)} = 1 - \mu \int_0^1 \frac{\psi(x) H(x) dx}{\mu + x},$$

where $\psi(x)$ is non-negative for $0 \leq x \leq 1$ and satisfies the condition

$$\int_0^1 \psi(x) dx \leq \tfrac{1}{2}.$$

It is assumed that the characteristic function is regular for a domain D of the z-plane, which includes every point of the closed interval $(-1, 1)$. It is proved that the solution H is given by a definite integral expression in the complex z-plane. A unique class of further solutions is found from H. *M. J. O. Strutt* (Zürich)

FUNCTIONAL ANALYSIS

See also 6377, 6468, 6501, 6502, 6503, 6568, 6623, 6624, 6625, 6626.

6642:

Raĭkov, D. A. Completely continuous spectra of locally convex spaces. Trudy Moskov. Mat. Obšč. **7** (1958), 413–438. (Russian)

Let X, Y, X_α, $\cdots$ denote locally convex topological linear spaces over the complex numbers. A set $A \subset X$ totally bounds a set $B \subset X$ if for every $\varepsilon > 0$ there is a finite set $F_\varepsilon \subset B$ such that $B \subset \varepsilon A + F_\varepsilon$. "Totally bounded by a neighborhood of 0" is abbreviated "totally bounded". A linear mapping $X \to Y$ is called bounded [or, respectively, totally bounded, totally continuous, or (bi)compact] if the image of some neighborhood of 0 in X is bounded [or, respectively, is totally bounded, has compact closure, or is compact]. An inverse linear-mapping system $\{\pi_\alpha{}^\beta : \pi_\alpha{}^\beta X_\beta \subset X_\alpha,\ \alpha < \beta\}$ [respectively, a direct mapping system $\{\pi_\beta{}^\alpha : \pi_\beta{}^\alpha X_\alpha \subset X_\beta,\ \alpha < \beta\}$] is totally continuous if for each α there is a $\beta > \alpha$ such that $\pi_\alpha{}^\beta$ [resp., $\pi_\beta{}^\alpha$] is totally continuous. The limit-space X is a totally continuous spectrum. In the direct case, an injective system ($\pi_\beta{}^\alpha$ injections) can be used to obtain the same limit. A space X is of type (S) (for 'Schwartz') [A. Grothendieck, Summa Brasil. Math. **3** (1957), 57–123; MR **17**, 765] if each neighborhood of 0 totally bounds some other neighborhood of 0. Proofs of some of Grothendieck's basic results are provided because only their statements were available to the author at the time of writing. A totally continuous inverse spectrum X is called an $(\bar{S})$ space. $(\bar{S})$ spaces are shown (§ 5) to be complete (S) spaces; and each complete (S) space is of type $(\bar{S})$. The metrizable $(\bar{S})$ spaces are discovered to be precisely the (M^*) spaces of Sebastião e Silva [Rend. Mat. e Appl. (5) **14** (1955), 338–410; MR **16**, 1122]. A space of type $(\bar{S}')$ is a totally continuous direct spectrum. Every such space is representable as an injective limit of Banach spaces with compact injections. The main result (§ 6) appears to be that the class $(\bar{S}')$ coincides with the class of inductive limits of Banach spaces. If $X \in (\bar{S}')$ and $Y \subset X$ is closed, then $X/Y \in (\bar{S}')$. Turning to duality, the author shows that for $X \in (\bar{S})$, the strong dual (X', b) is of type $(\bar{S}')$, and indeed the entire class $(\bar{S}')$ can be obtained in this manner. However, the strong dual of an $(\bar{S}')$ space X is of type $(\bar{S})$ (if and) only if each bounded set in X is totally bounded by some absolutely convex (i.e., $\lambda K + \mu K \in K$ for $|\lambda| + |\mu| \leqq 1$) bounded set K. § 8 generalizes Schauder's theorem on the compactness of the conjugate of a totally continuous operator. The final § 10 investigates inverse limits X which project densely on the various X_α.

R. Arens (Los Angeles, Callf.)

6643:

Alexiewicz, A.; and Semadeni, Z. A generalization of two norm spaces. Linear functionals. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 135–139.

The authors state the following special case of Grothendieck's theorem [N. Bourbaki, *Espaces vectoriels topologiques*, Actualités Sci. Ind. no. 1230, Hermann, Paris, 1955; MR **17**, 1109; Chap. IV, Section 3, Ex. 3]: Let X be a locally convex topological vector space, and S a convex, équilibrée subset which is closed, bounded, and absorbing; then the completion of the dual space X' under the norm defined by the polar S^0 consists of those linear functionals on X which are continuous on S. The proof of this is given essentially in an accompanying paper [#6644 below].

As applications they show (A) that certain linear functionals on the Banach space of bounded continuous functions on a completely regular Hausdorff space are given by Borel measures, and (B) that a linear functional on a Banach space is continuous if and only if it is weakly continuous on the unit ball. The third, and principal, application is as follows. Suppose that X and S are as above; that in addition X is a vector lattice; S is normal in the sense that $x \in S$, $|y| \leqq |x| \to y \in S$; and the topology of X has a neighborhood basis at 0 consisting of normal sets; that finally X_0 is a normal linear subspace. Then a linear functional on X_0 which is continuous on S can be extended to a linear functional on X with the same property. *S. Kaplan* (Detroit, Mich.)

6644:

Alexiewicz, A.; and Semadeni, Z. Linear functionals on two-norm spaces. Studia Math. **17** (1958), 121–140.

This note corrects an error in a preceding paper of the first author, and then goes on to show that, while there do exist locally convex topologies in which the convergent sequences are just the two-norm convergent sequences, it still need not happen that the extension property holds for two-norm continuous linear functionals.

M. M. Day (Urbana, Ill.)

6645:

Veksler, A. I. On the Archimedean principle in semiordered factor lineals. Dokl. Akad. Nauk SSSR **121** (1958), 775–777. (Russian)

Kantorovič, Vulih, and Pinsker [*Funkcional'nyĭ analiz v poluuporyadočennyh prostranstvah*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950; MR **12**, 340] defined a general vector lattice which they called a K-lineal; a complete K-lineal was called a K-space. The present paper announces some half dozen theorems; of these, one gives a necessary and sufficient condition that a factor space derived from an Archimedean ordered K-lineal be again Archimedean ordered; a second theorem takes care of the K-space case; and the remaining theorems deal with particular types of K-spaces.

R. M. Baer (Berkeley, Calif.)

6646:

Bauer, Heinz. Über die Fortsetzung positiver Linearformen. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1957**, 177–190 (1958).

In an article previously reviewed in detail [C. R. Acad. Sci. Paris **244** (1957), 289–292; MR **18**, 660] the author stated theorems on extensions of positive linear functionals on topological vector spaces. These are here proved, and applications are given. Among these is a proof that in a normed space a functional on a subspace which possesses an extension as a continuous positive linear functional to the whole space possesses at least one such extension whose norm is minimal. [Cf. #6468 above.]

J. L. B. Cooper (Cardiff)

6647:

Alexiewicz, A. Addition to the paper "On some theorems of S. Saks". Studia Math. **17** (1958), 69.

The author corrects Theorem 3, p. 25 of the paper

quoted [Studia Math. 13 (1953), 18–29; MR 14, 1070] to include in the hypotheses of the theorem the statement that there exists a $\rho>0$ depending on ε such that the stated conclusions hold.

R. E. Fullerton (College Park, Md.)

6648:

Ringrose, J. R. A note on uniformly convex spaces. J. London Math. Soc. 34 (1959), 92.

A short proof of the well-known theorem that a uniformly convex Banach space is reflexive.

D. C. Kleinecke (Livermore, Calif.)

6649:

Mityagin, B. S. On the second mixed derivative. Dokl. Akad. Nauk SSSR 123 (1958), 606–609. (Russian)

Let E be a Banach space of functions $f(s, t)$ defined on a two-dimensional torus such that the set D^∞ of all infinitely differentiable functions is dense in E. S. N. Bernšteĭn [*Sobranie sočinenii*, Tom I, Izdat. Akad. Nauk SSSR, Moscow, 1952; MR 14, 2; page 97] proved that in the cases when E is either L^2 or the space W of all functions with absolutely convergent Fourier series the following is true: if $f(s, t)$ is a continuous function such that the second derivatives f_{ss} and f_{tt} belong to E, then the mixed derivative f_{st} also belongs to E.

The author shows, giving a counter example, that the above statement is false if E is the space C of all continuous functions defined on the torus.

P. Saworotnow (Washington, D.C.)

6650:

Mrówka, S. Functionals on uniformly closed rings of continuous functions. Fund. Math. 46 (1958), 81–87.

Let R be an algebra of real-valued continuous functions on a completely regular space X containing the constant functions and closed with respect to uniform convergence. Conditions are given under which (*) each non-trivial multiplicative linear functional α on R is of the form $\alpha(f)=f(p_0)$ for some $p_0\in X$.

Let R^* be the set of all $f\in R$, $0\leq f(p)\leq 1$, $p\in X$. Let K be the cartesian product of unit intervals [0, 1], one factor for each $f\in R^*$. Define a mapping F_R of X into K by the rule that, for $p\in X$, $F_R(p)$ has $f(p)$ as the coordinate value for the coordinate corresponding to $f\in R^*$. Theorem 1. If R consists entirely of bounded functions, then (*) holds if and only if $F_R(X)$ is compact.

A subset P of a topological space is called Q-closed in the space if for each $p\notin P$ there is a G_δ-set containing p and disjoint from P. Theorem 2. Suppose R has the property that, for each $f\in R$ which is nowhere zero, $1/f\in R$. Then (*) holds if and only if $F_R(X)$ is Q-closed in $\overline{F_R(X)}$. The condition on Q-closure holds if X is a Lindelöf space.

B. Yood (New Haven, Conn.)

6651:

Tihonov, A. N.; and Samarskiĭ, A. A. The representation of linear functionals in the class of discontinuous functions. Dokl. Akad. Nauk SSSR 122 (1958), 188–191. (Russian)

This note represents an arbitrary bounded linear functional A on the space of all piecewise continuous functions on an interval $[a, b]$. The representation may be written in the (abbreviated) form

$$A(f)=\int_a^b f(x)da(x)+\sum_{j=1}^\infty \{\alpha_jf(x_j-0)+\beta_jf(x_j)+\gamma_jf(x_j+0)\},$$

where a is a continuous function of bounded variation, (x_j) is a sequence of numbers in the interval and the α, β, and γ are numbers that can be identified in more detail.

R. G. Bartle (Urbana, Ill.)

6652:

Gapoškin, V. F. On unconditional bases in L^p ($p>1$) spaces. Uspehi Mat. Nauk 13 (1958), no. 4(82), 179–184. (Russian)

This note gives a necessary and sufficient condition that a basis $\{f_k\}$ in $L^p(a, b)$ ($p>1$) be an unconditional basis. It is that (i) for any f in L^p the coefficients $\{c_k\}$ satisfy

$$(*)\qquad \left\{\int_a^b\left(\sum_{k=1}^\infty a_k^2f_k^2(x)\right)^{p/2}dx\right\}^{1/p}<\infty,$$

and (ii) any numerical sequence $\{a_k\}$ satisfying (*) is the sequence of coefficients of some function in L^p. When the condition holds, the expression (*) gives a norm equivalent to the usual norm in L^p. It is also proved that if $p>1$, $p\neq 2$, then no normalized unconditional basis in L^p can be a uniformly bounded set of functions.

R. G. Bartle (Urbana, Ill.)

6653:

Ding, Shia-Shi. On an imbedding theorem. Sci. Record (N.S.) 1 (1957), 315–318. (Russian)

Let $\Phi(u)$, $X(u)$ be two monotone convex functions, with $X(u)\leq C\Phi(u)$ for $u\geq u_0$, which satisfy certain regularity conditions. One can take in particular $X=u^q$, $\Phi=u^p$, $1\leq q\leq p$. If $\Psi=\Phi X^{-1}$, if Ψ_1 is the Young conjugate of Ψ, and if

$$\int_0^1\Psi_1(r^{-n-1}X(r^{-l})^{-1}\Psi^{-1}(r^{-n-1})^{-1})dr<+\infty,$$

then $D_\Phi{}^l\subset L_X$. Here, L_X is an Orlicz space of functions $f(x_1, \cdots, x_n)$ on the n-dimensional space and $D_\Phi{}^l$ is the subset of L_Φ with $(x_1^2+\cdots+x_n^2)^{l/2}|f|\in L_\Phi$.

G. G. Lorentz (Syracuse, N.Y.)

6654:

Ringrose, J. R. Operators of Volterra type. J. London Math. Soc. 33 (1958), 418–424.

Let B be a complex Banach space. A family $\{E_\lambda\}$ of continuous projections in B defined for λ in [0, 1] is called here a continuous resolution of the identity if the following conditions are satisfied. (1) $E_0=0$, $E_1=I$.

$$(2)\qquad E_\lambda E_\mu=E_\mu E_\lambda=E_\lambda\ (0\leq\lambda\leq\mu\leq 1).$$

(3) E_λ is continuous in the sense that, for each $x\in B$, $\|E_\lambda x-E_\mu x\|\to 0$ as $\mu\to\lambda$. A bounded linear operator on B is called a Volterra operator if (a) there is a continuous resolution of the identity $\{E_\lambda\}$, where $TE_\lambda=E_\lambda TE_\lambda$, $0\leq\lambda\leq 1$, and (b) T is asymptotically quasi-compact, that is, there exists a sequence $\{K_n\}$ of compact linear operators on B such that $\|T^n-K_n\|^{1/n}\to 0$. It is shown that any Volterra operator T on B is quasi-nilpotent. The author had previously shown this for T compact [Proc. Cambridge Philos. Soc. 51 (1955), 44–55; MR 16, 716]. An example is given of a bounded linear operator on Hilbert space which is compact and quasi-nilpotent but is not a Volterra operator.

B. Yood (New Haven, Conn.)

6655:

Dikiĭ, L. A. Trace formulas for Sturm-Liouville differential operators. Uspehi Mat. Nauk (N.S.) 13 (1958), no. 3(81), 111–143. (Russian)

Let λ_n be the eigen-values of $-u''+p(x)u=\lambda_u$, $u(0)=u(\pi)=0$. From the identity

$$(*)\qquad \sum(\lambda_n+\zeta)^{-1}=\mathrm{Sp}(-d^2/dx^2+p(x)+\zeta)^{-1}$$

one can obtain the formulae

(**) $\sum \lambda_n^k = \mathrm{Sp}(-d^2/dx^2+p(x))^k$ $(k=1, 2, \cdots)$

by expanding formally in powers of ζ^{-1} and equating coefficients. The apparently meaningless results (**) can be "regularised" by forming rigorous asymptotic expansions of both sides of (*) for large ζ, assuming $p(x)$ to be infinitely differentiable. For the left side of (*) this was done by I. M. Gel'fand [same Uspehi **11** (1956), no. 1(67), 191–198; MR **18**, 129], whose argument is reproduced in the present survey. An asymptotic expansion is also found for the right side of (*), assuming that all odd-order derivatives of $p(x)$ vanish for $x=0, \pi$. In particular, the author finds that

$$\sum_{n=1}^{\infty}(\lambda_n-n^2)=-\tfrac{1}{4}(p(0)+p(\pi))$$

if $\int_0^\pi p(x)dx=0$ [cf. I. M. Gel'fand and B. M. Levitan, Dokl. Akad. Nauk SSSR **88** (1953), 593–596; MR **15**, 33]; analogous formulae are given concerning $\sum \lambda_n^2$, $\sum \lambda_n^3$, a numerical application being worked. The connection with perturbation procedures is also examined.

F. V. Atkinson (Canberra City)

6656:

Tautz, Georg. Beiträge zur Störungstheorie. Arch. Math. **9** (1958), 287–296.

This note extends some earlier work of the author [same Arch. **5** (1954), 401–413, **7** (1956), 310–316; MR **16**, 371; **18**, 661]. (I) It is proved that if (K_n) is a sequence of bounded linear operators in a Banach space which converges in norm to K_0 and if the complex number λ_n is in the boundary of the spectrum $\sigma(K_n)$, $n\geqq 1$, and (λ_n) converges to λ_0, then λ_0 is in $\sigma(K_0)$. {Reviewer's remark. It is sufficient that λ_n be in $\sigma(K_n)$.} (II) Conversely, let $\sigma(K_0)$ have property (A) at λ_0: there exists a sequence of Jordan curves in $\rho(K_0)$ surrounding λ_0 and contracting to λ_0. (This condition holds if $\sigma(K_0)$ has local capacity zero at λ_0.) Then, if $K_n\to K$ and λ_0 is in $\sigma(K_0)$, there are λ_n in $\sigma(K_n)$ such that $\lambda_n\to\lambda_0$. (III) If K_0 belongs to a compact set $\mathfrak{K}$ of bounded operators, each with property (A) at every point of their spectra, and if λ_0 is in $\sigma(K_0)$, then every operator in $\mathfrak{K}$ near K_0 has a spectral point near λ_0. (IV) A perturbation procedure employed by P. C. Rosenbloom [ibid. **6** (1955), 89–101; MR **16**, 832] is used to find a condition sufficient to insure that if K is near K_0 and K_0 has a real eigenvalue λ_0, then K has a real eigenvalue near λ_0.

R. G. Bartle (Urbana, Ill.)

6657:

Lin', Cyun'. L. V. Kantorovič's theory of approximation methods in analysis. Sci. Record (N.S.) **2** (1958), 92–97. (Russian)

Consider the equation

(1) $$Kx\equiv x-\lambda Hx=y,$$

in which the given y and the unknown x are elements of a normed linear space X, while λ is a scalar, and H denotes an operator mapping X into X. Beside (1), the "approximating equation"

(2) $$\bar{K}\bar{x}\equiv\bar{x}-\varphi\lambda\bar{H}\bar{x}=\varphi y$$

is considered. Here $\bar{x}$ is an element of a normed linear space $\bar{X}$, while $\bar{H}$ and φ are operators mapping $\bar{X}$ into X and X into $\bar{X}$, respectively. The author states the following condition I): if $\bar{x}_0$ is a solution of (2), i.e., of $\bar{x}_0=\varphi(\lambda\bar{H}\bar{x}_0+y)$, then the existence of three positive constants ε_1, ε_2, ε_3 is assumed such that

$$\|H(\lambda\bar{H}\bar{x}_0+y)-\bar{H}\varphi(\lambda\bar{H}\bar{x}_0+y)\|\leq\varepsilon_1\|\lambda\bar{H}\bar{x}_0+y\|+\varepsilon_2\|\bar{x}_0\|+\varepsilon_3.$$

The following theorem I is then stated (without proof). Assume that $\bar{X}$ is complete, that $\varphi\bar{H}$ is completely continuous, that condition I) is satisfied, that the inverse K^{-1} of K exists and that $\|K^{-1}\|$ satisfies a certain inequality involving λ, $\|\varphi\|$, ε_1, ε_2. Then $\bar{K}^{-1}$ also exists, and the norm of

(3) $$x^*-(\lambda\bar{H}\bar{x}_0+y),$$

where x^* is a solution of (1), satisfies a certain inequality involving the ε_i, $\|K^{-1}\|$, λ, $\|x^*\|$, $\|\varphi\|$, and showing, in particular, that (3)$\to 0$ as the $\varepsilon_i\to 0$. The author states another condition, and a corresponding theorem II, asserting, essentially, that under this second condition the existence of $\bar{K}^{-1}$ (together with a certain boundedness condition on $\|\bar{K}^{-1}\|$) implies the existence of K^{-1}. Again an estimate for (3) is given.

It is then shown that, for quite a number of approximation procedures for solutions of Fredholm integral equations, proofs for convergence and estimates for the error may be obtained by proper specialization of the above theorems.

Finally, equation (1) is generalized to an equation of the form

$$Kx\equiv Gx-Tx=y,$$

where $x\subset X$ and where y is in some other normed space Y while G and T are mappings of X into Y. With a corresponding generalization of the approximating equation (2), an analogue of theorem I is stated, and an application to a boundary value problem of a linear ordinary differential equation of even order is made.

E. H. Rothe (Ann Arbor, Mich.)

6658:

Halmos, Paul R.; and Kakutani, Shizuo. Products of symmetries. Bull. Amer. Math. Soc. **64** (1958), 77–78.

A (bounded) operator Q on a (complex) Hilbert space H is a symmetry if $Q^*Q=QQ^*=Q^2=I$. The authors prove that, if H is infinite dimensional, every unitary operator on H is the product of four symmetries, and there exists a unitary operator on H which is not the product of three symmetries.

R. E. Fullerton (College Park, Md.)

6659:

Putnam, C. R. Commutators and absolutely continuous operators. Trans. Amer. Math. Soc. **87** (1958), 513–525.

Let A and B be bounded linear operators on a Hilbert space H and define

$$C=B^{(1)}=AB-BA,$$
$$D=B^{(2)}=AC-CA,$$
$$E=B^{(3)}=AD-DA.$$

Let W_C be the convex closure of the set of complex numbers $\{(Cx, x)\}$ for $x\in H$, $\|x\|=1$, and let W_D, W_E be similarly defined. Assume A normal with spectral resolution $\int z dK$. Continuing investigations initiated in other papers [Amer. J. Math. **73** (1951), 127–131; Proc. Amer. Math. Soc. **7** (1956), 1026–1030; MR **12**, 836; **18**, 495] the author investigates the relationship between W_C, W_D, or W_E containing 0 as an interior point (relative to the dimension of the set) and the value of $\int_Z dK$ for the set Z of measure zero in various dimensions. In particular, if 0 is not interior to W_D, then $\int_Z dK<I$ for all Z of zero two-dimensional measure. If A is normal, 0 is always interior to W_E. Conditions in terms of W_C $[W_D]$ are given

which insure that $\int_Z dK=0$ for A self-adjoint [normal] for all sets Z of one-[two-]dimensional measure zero. These theorems are applied to the investigation of properties of $\int_Z dK$ for Toeplitz matrices (matrices of the form (c_{i-j}) for c_k Fourier coefficients). For certain types of these matrice it iss shown that $\int_Z dK=0$ for all Z of one-dimensional measure zero.

R. E. Fullerton (College Park, Md.)

6660:

Putnam, C. R. On the numerical ranges of commutators. J. London Math. Soc. **34** (1959), 23–26.

Suppose $C=AB-BA$ and $D=A^*B-BA^*$. Let W_C be the numerical range of C, that is, the set of numbers (Cx, x) where $\|x\|=1$; and similarly W_D. Several relationships between W_C and W_D are demonstrated; for example, the sets have a non-zero intersection.

D. C. Kleinecke (Livermore, Calif.)

6661:

Castoldi, Luigi. Operatori hermitiani anticommutabili. Rend. Sem. Fac. Sci. Univ. Cagliari **27** (1957), 35–44.

This paper considers two "complete" Hermitian operators, A and B, in Hilbert Space. These operators anticommute, that is, $AB+BA=0$. The fundamental result is most easily described in the case in which A has a simple point spectrum. Then the characteristic vectors of A occur in pairs, ϕ_λ and $\phi_{-\lambda}$, such that

$$A\phi_\lambda=\lambda\phi_\lambda,\ A\phi_{-\lambda}=-\lambda\phi_{-\lambda},\ B\phi_\lambda=\beta\phi_{-\lambda},\ B\phi_{-\lambda}=\bar\beta\phi_\lambda$$

for a complex β. The term "complete" as applied to B must be taken to rule out the possibility that $\beta=0$. Since the manifold determined by ϕ_λ and $\phi_{-\lambda}$ reduces B, two corresponding characteristic vectors for B are determined.

The author applies this result to characterize a transformation L which has the property that there exists a Hermitian transformation X such that XL is Hermitian.

F. J. Murray (New York, N.Y.)

6662:

Allen, H. S. Rings of infinite matrices. Quart. J. Math. Oxford Ser. (2) **8** (1957), 117–118.

The author makes the remark (in the terminology of Köthe and Toeplitz [J. Reine Angew. Math. **171** (1934), 193–226]) that the set of all matrices which map a Köthe-Toeplitz sequence space α into itself is a ring if $\beta\subseteq\alpha\subseteq\beta^{**}$, and the space β is normal and contains all finite sequences. *G. G. Lorentz* (Syracuse, N.Y.)

6663:

Gendler, M. G. On one parameter groups of functional transformations. Dokl. Akad. Nauk SSSR (N.S.) **111** (1956), 524–527. (Russian)

Let F be the class of continuous functions f on the real axis, such that $f=O(\exp \varepsilon x^2)$, $\varepsilon>0$. Let L_u be the transformation acting in F such that

$$(L_uf)(x)=p(u)\int_{-\infty}^{\infty} f(y)\ \exp[-a(u)x^2+2b(u)xy-c(u)y^2]dy,$$

where a, b, c, p are continuous, and a, $c>0$.

Suppose that $L_uL_v=L_{uv}$ $(u, v>1)$. The author shows that this semi-group of operators is equivalent to one of the following two:

$$(A_uf)(x)=u^\beta(2\pi k\ln u)^{-\frac12}\int f(y)\exp\left\{-\frac{(x-y)^2}{2k\ln u}\right\}dy,$$

$$(B_uf)(x)=u^{\beta+\alpha/2}[2\pi c(u^{2\alpha}-1)]^{-\frac12}$$
$$\times\int f(y)\exp\left\{\frac{-[(u^{2\alpha}+1)(x^2+y^2)-4u^\alpha xy]}{4c(u^{2\alpha}-1)}\right\}dy.$$

These two, essentially, have been studied by N. P. Romanoff [Ann. of Math. (2) **48** (1947), 216–233; MR **8**, 520]. The author then examines the question of extending such a semigroup to a group acting in the space of entire functions $O(\exp \varepsilon|z|^2)$, $\varepsilon>0$.

R. Arens (Los Angeles, Calif.)

6664:

Bharucha-Reid, A. T. Ergodic projections for semi-groups of periodic operators. Studia Math. **17** (1958), 189–197.

The author proves the following theorem: Let T be a bounded linear operator in a B-space which has the property that $T^{\omega+1}=T$ for some integer $\omega>0$. Then

$$\lim_{n\to\infty} n^{-1}\sum_{i=1}^{n} T^i=\omega^{-1}\sum_{i=1}^{\omega} T^i.$$

In doing this, he relies on the ergodic theorem of K. Yosida and S. Kakutani [Ann. of Math. (2) **42** (1941), 188–228; MR **2**, 230] for the existence of the limit and proves that the expression above is its value.

He then applies this result to the case of a periodic semi-group of operators indexed by the real numbers in the standard way [cf. Yosida and Kakutani, op. cit.].

Continuing in this line, he applies his theorem to Markoff processes with countably many states which are stationary and obey $T_{\omega+s}(a_i, a_j)=T_s(a_i, a_j)$ for some $\omega>0$ and for all $s>0$, and shows that the long-term average is exactly the average between 0 and ω.

A. Beck (Madison, Wis.)

6665:

Tomiyama, Jun. A remark on the invariants of W^*-Algebras. Tôhoku Math. J. (2) **10** (1958), 37–41.

Methods similar to those of Dixmier [C. R. Acad. Sci. Paris **238** (1954), 439–441; MR **15**, 721] are employed to study spatial isomorphisms of W^*-algebras. Theorem 3: Let M_1 and M_2 be W^*-algebras which do not contain such components as $(\mathrm{II}_\infty, \mathrm{II}_1)$ and $(\mathrm{II}_1, \mathrm{II}_1)$. Suppose that θ_1, θ_2 are isomorphisms between M_1 and M_2, M_1' and M_2', respectively. If θ_1 and θ_2 coincide on the center of M_1, then θ_1 is spatial. These results are similar to those of Pallu de la Barrière [Bull. Soc. Math. France **82** (1954), 1–52; MR **16**, 491].

E. L. Griffin, Jr. (Ann Arbor, Mich.)

6666:

Takesaki, Masamichi. A note on the cross-norm of the direct product of operator algebra. Kōdai Math. Sem. Rep. **10** (1958), 137–140.

Let M_1 and M_2 be two C^*-algebras, and let $M_1\odot M_2$ be their algebraic tensor product. One way to define a norm on $M_1\odot M_2$ is to consider M_1 and M_2 as C^*-algebras of operators on two Hilbert spaces and to take the norm of an element of $M_1\odot M_2$ to be its norm as an operator on the tensor product of the two Hilbert spaces. Another way to norm $M_1\odot M_2$ is as bilinear functionals on $M_1^*\times M_2^*$, * meaning conjugate as a Banach space. This note shows that these two norms are equal if and only if one of the C^*-algebras is commutative.

W. F. Stinespring (Princeton, N.J.)

6667:

Helgason, Sigurdur. Lacunary Fourier series on non-commutative groups. Proc. Amer. Math. Soc. **9** (1958), 782–790.

The author finds an extension of the concept of lacunarity for classical Fourier series to series over a compact group G. This is achieved by means of a "hypergroup" structure introduced on $\hat G$, the set of Peter-Weyl repre-

sentations of G. Theorems of Kolmogoroff and Banach are generalized; as a specimen, theorem 5 states that if $f \in L^1(G)$ is central and has a (generalized) lacunary Fourier series, then $f \in L^2(G)$.

J. G. Wendel (Ann Arbor, Mich.)

6668:

Kunze, R. A. L_p Fourier transforms on locally compact unimodular groups. Trans. Amer. Math. Soc. 89 (1958), 519–540.

The Hausdorff-Young theorem asserts that the $L_{p'}$-norm of a function in L_1 and L_p of a locally compact abelian group ($1<p\leqq 2$) is greater than or equal to the $L_{p'}$-norm of its Fourier transform, where $1/p+1/p'=1$. This paper extends this theorem to locally compact unimodular groups. The harmonic analysis used is purely global. That is, the Fourier transform of a function in L_p ($1\leqq p\leqq 2$) of a unimodular group G is defined as the operator which is left convolution by the function acting on $L_2(G)$. This operator is shown to be measurable in the sense of I. E. Segal [Ann. of Math. (2) 57 (1953), 401–457; MR 14, 991] relative to the von Neumann algebra generated by left translations. The canonical gage space of G is this algebra equipped with the gage in the sense of Segal [op. cit.] that makes the L_2-norm of a function on G equal to the L_2-norm of its Fourier transform. The author discusses L_p-norms in gage spaces, and then shows that the $L_{p'}$-norm of a function in $L_p(G)$ ($1<p\leqq 2$) exceeds or equals the $L_{p'}$-norm of its Fourier transform. Also, he shows that if a function f has its Fourier transform in L_p of the canonical gage space ($1<p\leqq 2$), then the $L_{p'}$-norm of f is less than or equal to the L_p-norm of its Fourier transform. The method of proof is by establishing a Riesz-Thorin interpolation theorem for gage spaces.

W. F. Stinespring (Princeton, N.J.)

6669:

Fischer, H. R. Differentialkalkül für nicht-metrische Strukturen. II: Differentialformen. Arch. Math. 8 (1957), 428–443.

Continuing the work of his earlier paper [Ann. Acad. Sci. Fenn. Ser. A.I. no. 247 (1957); MR 19, 869] on the differentiation of mappings of one locally convex topological vector space into another, the author gives the corresponding theory of differential forms and their integration over differentiable simplices, proving the lemma $d(df)=0$ and its converse, and proving Stokes' theorem.

J. T. Schwartz (New York, N.Y.)

CALCULUS OF VARIATIONS

See also 6489.

6670:

Hašaev, A. H. Semi-continuity and absolute minimum in the simplest problem of the calculus of variations. Mat. Sb. N. S. 45(87) (1958), 423–432. (Russian)

Another proof is given for a theorem of Tonelli [Fondamenti di calcolo delle variazioni, Zanichelli, Bologna, 1921–1923] which asserts the existence of an absolute minimum if the integrand $f(x, y, y')$ is semi-regular and $f(x, y, y')\geqq \alpha y'^2+\beta$; α, β positive constants.

W. H. Fleming (Providence, R.I.)

GEOMETRIES, EUCLIDEAN AND OTHER

See also 6423, 6424, 6443.

6671:

★Forder, Henry George. The foundations of Euclidean geometry. Dover Publications, Inc., New York, 1958. xiii+349 pp. $2.00.

An unabridged and unaltered republication of the first edition [University Press, Cambridge, 1927].

6672:

★Sommerville, D. M. Y. An introduction to the geometry of n dimensions. Dover Publications, Inc., New York, 1958. xviii+196 pp. $1.50.

An unabridged and unaltered republication of the first edition [Methuen, London, 1929].

6673:

Marmion, A. Sur les axes des cylindres de révolution passant par 2, 3, 4, 5 points. Mathesis 66 (1957), 261–268.

L'objet de cet article est d'étudier les différentes figures géométriques définies par les axes des cylindres de révolution qui passent par 2, 3, 4, 5 points donnés. En voici les résultats essentiels.

Le complexe des axes Δ_{12} des cylindres de révolution passant par deux points donnés A_1, A_2 est un complexe tétraédral harmonique par rapport au tétraèdre ayant pour faces le plan médiateur de A_1A_2, le plan de l'infini et les deux plans isotropes issus de A_1A_2.

La congruence des axes Δ_{123} des cylindres de révolution passant par trois points donnés A_1, A_2, A_3 est une congruence du quatrième ordre et de la troisième classe.

La surface réglée engendrée par les axes Δ_{1234} des cylindres de révolution passant par quatre points donnés A_1, A_2, A_3, A_4 est du neuvième ordre.

Enfin il y a six cylindres de révolution qui passent par cinq points donnés A_1, A_2, A_3, A_4, A_5. Les six axes de ces cylindres de révolution sont parallèles aux génératrices d'un cône du second ordre.

F. Şemin (Istanbul)

6674:

Iorga, Mircea. Nouveaux procédés pour l'identification de l'ordre de succession des sommets dans les polygones d'intersection de deux polyèdres. Bul. Inst. Politehn. Bucureşti 19 (1957), no. 3/4, 55–67. (Romanian. Russian and French summaries)

Two procedures are described for finding the order of the vertices on the polygons occuring in the intersection of two polyhedra. The problem is posed and solved in the terms of descriptive geometry.

H. W. Kuhn (London)

6675:

Griffin, John S., Jr.; and McLaughlin, J. E. A theorem on two-dimensional vector spaces. Michigan Math. J. 4 (1957), 257–259.

Let V be a two-dimensional vector space over a division ring D; let Π_V be the family of all lines of V which pass through the origin.

The following theorem is proved. Let V and W be two-dimensional vector spaces over division rings D and E, respectively, and suppose f: $\Pi_V\to\Pi_W$ is one-to-one onto. Suppose further that if G and H denote the respective projective groups, then the map $f^*: G\to H$ given by $pf^*=f^{-1}pf$ is an isomorphism. Then either D is isomorphic to E, or D is anti-isomorphic to E.

From the introduction

6676:

Magari, Roberto. Le configurazioni parziali chiuse contenute nel piano, P, sul quasicorpo associativo di ordine 9. Boll. Un. Mat. Ital. (3) 13 (1958), 128–140.

Let F_9 be a near-field of order 9 and let P be the projective plane on F_9 (nine is the best possible order for non-desarguesian planes). This note is concerned with the determination of the totality of the non-degenerate subplanes of P: P contains such planes of order two and three only. Those of order two are all collinear and those of order three are divided into two projectively different types. *F. Gherardelli* (Florence)

6677:

André, Johannes. Über Perspektivitäten in endlichen affinen Ebenen. Arch. Math. 9 (1958), 228–235.

A perspectivity of an affine plane is a collineation mapping lines onto themselves or onto parallel lines. Translations are perspectivities leaving no finite point fixed. The author assumes that there are non-trivial translations in at least two distinct directions. It is known that in this case the translations of a finite plane form an elementary Abelian group T. The translations $T(L)$ leaving a line L fixed form a group. The endomorphisms of $T(L)\neq 1$ form a Galois-field, K. If π is the group of all perspectivities, there will be one of the transitive constituents U of π, as a permutation group of points, which is also a transitive constituent for T. The group π when restricted to U is called the "Streckungsgruppe" S. It is shown that S is isomorphic to the multiplicative group of a subfield K^* of K. It is shown that π cannot have exactly two transitive constituents.

Marshall Hall, Jr. (Columbus, Ohio)

6678:

Kustaanheimo, Paul. On the relation of order in geometries over a Galois field. Soc. Sci. Fenn. Comment. Phys.-Math. 20 (1957), no. 8, 9 pp.

Order in a finite geometry over GF(p^n) satisfying the Pasch axioms depends on regarding quadratic residues as positive when p is odd. But for $p=2$ the Pasch axioms cannot hold. Weaker forms of the Pasch axioms can be satisfied in a variety of ways when $p=2$.

Marshall Hall, Jr. (Columbus, Ohio)

6679:

★Sommerville, D. M. Y. The elements of non-euclidean geometry. Dover Publications, Inc., New York, 1958. xvi+274 pp. $1.50.

An unabridged and unaltered republication of the first edition [Bell, London, 1914].

6680:

Bottema, O. On the medians of a triangle in hyperbolic geometry. Canad. J. Math. 10 (1958), 502–506.

In non-euclidean geometry let G_i be the internal point of the side opposite the vertex A_i ($i=1, 2, 3$) of a triangle and let G be the common point of the lines A_iG_i. Restricting himself to hyperbolic geometry, the author makes some remarks on the ratio GG_i/A_iG_i and proves the following result: In a triangle either all three ratios GG_i/A_iG_i are less than 1/3 or two of them are $<1/3$ and the third (belonging to the smallest side) is $\geqq 1/3$.

C. Longo (Parma)

CONVEX SETS AND DISTANCE GEOMETRIES

See also 6412, 6689, 6948.

6681:

Proskuryakov, I. V. A property of n-dimensional affine space connected with Helly's theorem. Uspehi Mat. Nauk 14 (1959), no. 1(85), 219–222. (Russian)

In n-dim. affine space a system of $n+2$ points can be decomposed into two disjoint non-empty subsystems whose convex closures have a point in common. The author proves this theorem and derives Helly's theorem in Radon's [Math. Ann. 83 (1921), 113–115] formulation, using König's [Math. Z. 14 (1922), 208–210] method for the infinite case. The following uniqueness theorem is also proved: the decomposition is unique if, and only if, for every $k=1, \cdots, n$, no $k+1$ points lie in a $k-1$ dim. plane. In this case the common point is unique, and the decomposition is determined by the condition: two points are in the same subsystem if they lie on the opposite sides of a plane through the remaining n points.

V. Linis (Ottawa, Ont.)

6682:

★Hannan, James. Approximation to Bayes risk in repeated play. Contributions to the theory of games, vol. 3, pp. 97–139. Annals of Mathematics Studies, no. 39. Princeton University Press, Princeton, N. J., 1957. $5.00.

6683:

Leuenberger, F. Einige Dreiecksungleichungen. Elem. Math. 13 (1958), 121–126.

Let ρ and r be the radii of the inscribed and circumscribed circles of a triangle having altitudes h_1, h_2, h_3. Then

$$9\rho\leqq h_1+h_2+h_3\leqq 9r/2.$$

The same inequality holds if the altitudes are replaced by the lengths of the medians or of the angle bisectors. Equality holds in the case of an equilateral triangle.

L. Moser (Edmonton, Alta.)

GENERAL TOPOLOGY, POINT SET THEORY

See also 6352, 6402, 6456, 6484, 6510, 6690.

6684:

Borsuk, K. Remarques sur la quasi-homéomorphie. Colloq. Math. 6 (1958), 1–4.

Two metric spaces X and Y are said to be quasi-homeomorphic when there exist for each $\varepsilon>0$ two continuous functions, ϕ from X to Y and ψ from Y to X, such that all the sets $\phi^{-1}(y)$, $y\in Y$, and $\psi^{-1}(x)$, $x\in X$, are of diameter less than ε. In answer to a question of T. Ganea, the author constructs in euclidean 3-space quasi-homeomorphic compact spaces X and Y such that X is homeomorphic to a subspace of the plane, but Y is not.

H. H. Corson (Seattle, Wash.)

6685:

Ponomarev, V. A new space of closed sets and many-valued mappings of bicompacts. Dokl. Akad. Nauk SSSR (N.S.) 118 (1958), 1081–1084. (Russian)

For any topological space X (with closed one-point sets), a space κX is introduced consisting of all non-void

closed $F \subset X$, with an open base made up of all $\{F | F \subset G\}$, G open in X; some properties of κX are stated. For compact X, Y, many-valued mappings f are considered (f assigns a closed set $fx \subset Y$ to every $x \in X$). Necessary and sufficient conditions are given for f to be continuous (i.e. such that for any $x \in X$ and any neighborhood V of fx there is a neighborhood U of x such that $fz \subset V$ whenever $z \in U$). Some of them are: (1) fA is closed whenever $A \subset X$ is closed, and $f'y = \{x | y \in fx\}$ is closed for any $y \in Y$; (2) $f'y$ is closed for any $y \in Y$, $\{f'y | y \in B\}$ is compact, as a subspace of κX, whenever $B \subset Y$ is closed. Various other results are given, most of them generalizing theorems well known for single-valued mappings. For some notions involved, cf. W. Strother [Duke Math. J. 22 (1955), 551–556; MR 17, 288]. *M. Katětov* (Prague)

6686:

Hulanicki, A. The completeness of the homeomorphisms group of a complete space. Colloq. Math. 5 (1958), 159–161.

Let a completely regular space X be complete in the finest uniform structure (compatible with the topology). The latter induces in the homeomorphism group $H(X)$ the structure u of uniform convergence; and u induces u^* in $H(X)$ through inversion. Then $H(X)$ is complete in $u \vee u^*$. *R. Arens* (Los Angeles, Calif.)

6687:

Reichbach, Marian. A topological theorem related to the theorem of Cantor-Bernstein. Riveon Lematematika 12 (1958), 27–30. (Hebrew. English summary)

"The main result of this paper is: If $A_1 \supset A_2 \supset A_3 = f(A_1)$ are compact sets such that A_2 is closed and open in A_1 and A_3 is closed and open in A_2, where f is any homeomorphism, and if the diameters of the sets $E, f(E), ff(E), \cdots$ tend to 0, where $E = A_2 \setminus A_3$, then the sets A_1 and A_2 are homeomorphic." (From the author's summary)

S. Eilenberg (New York, N.Y.)

6688:

Andrews, James J. A characterization of light open maps of Euclidean spaces into Euclidean spaces. Proc. Amer. Math. Soc. 9 (1958), 860–861.

A mapping f of R^n into R^m is said to be pseudo-monotone if for each closed set $X \subset R^m$ such that $R^m - X$ has no bounded component, $R^n - f^{-1}(X)$ also has no bounded component. It is shown that a light mapping of R^n into R^m is open if and only if it is pseudo-monotone.

E. Dyer (Chicago, Ill.)

6689:

Molnár, József. Über den zweidimensionalen topologischen Satz von Helly. Mat. Lapok 8 (1957), 108–114. (Hungarian. Russian and German summaries)

Let there be given a collection of simply connected regions in the plane; assume that the intersection of every two and every three of them is simply connected. A theorem of Helly then states that the intersection of all the regions is non-empty.

The author proves that the conclusion of the theorem still holds if we only assume that the intersection of every two regions is connected and the intersection of every three is non-empty. *P. Erdős* (Birmingham)

ALGEBRAIC TOPOLOGY

6690:

Stewart, T. E. On R-equivalent spaces. Nederl. Akad. Wetensch. Proc. Ser. A 61=Indag. Math. 20 (1958), 460–462.

The paper exhibits absolute neighborhood retracts X and Y, each homeomorphic to a retract of the other, such that $\pi_1(X) \not\approx \pi_1(Y)$. Both X and Y are countable unions of ordinary tori and arcs. $\pi_1(X)$ and $\pi_1(Y)$ are found explicitly. *M. E. Shanks* (Lafayette, Ind.)

6691:

Weier, Joseph. Eine Invariante sphärischer Abbildungen. Nederl. Akad. Wetensch. Proc. Ser. A. 60= Indag. Math. 19 (1957), 12–21.

Let S and T be two Euclidean spheres each having dimension greater than one, and let f be a continuous mapping from S into T. Denote by F the class of all continuous mappings f_1 from S into T each of which has the property that at no point p in S are the points $f(p)$ and $f_1(p)$ antipodal on T. Let a be a point of S and f_3, f_4 two mappings in F such that a is the only point of S where f_3 and f_4 have the same value — denote their common value at a by b. Let V be the set of all points of T whose distance from b is less than the radius of T. Then, if U is the set of all points of S whose distance from a is less than a sufficiently small positive number which is less than the radius of S, it is true that the images of $\bar{U}$ under both f_3 and f_4 are subsets of V. Let t be a topological mapping of $\bar{V}$ onto the closure $\bar{W}$ of a simplex W. For each point p in $\bar{U} - U$ let $h'(p)$ denote the point common to $\bar{W} - W$ and the ray from $t(b)$ through $t(b) + tf_3(p) - tf_4(p)$. The pair (f_3, f_4) is termed stable if and only if the mapping h' from the sphere $\bar{U} - U$ into $\bar{W} - W$ is essential. It is shown that there exist in F two mappings g_1 and g_2 which take on the same value at exactly one point of S and are arbitrarily close to f, and that every mapping g in F will take on the same value as f in at least one point of S if and only if the pair (g_1, g_2) is stable.

P. V. Reichelderfer (Columbus, Ohio)

6692:

Weier, Josef. Über Stabilität von Abbildungen. Arch. Math. 8 (1957), 340–346.

Let G be a homotopy class of continuous mappings from an $(n+1)$-sphere S into an n-sphere T, where n is not less than four. It is known [#6691 above] that for each f in G there exists a f' in G such that the distance between $f(p)$ and $f'(p)$ is always less than the diameter of T and is zero at precisely one point p in S. It is shown that if the pair (f, f') is stable [#6691] for one f in G, then it is stable for every f in G. *P. V. Reichelderfer* (Columbus, Ohio)

6693:

Weier, Joseph. Bemerkungen zu einer Note über stetige Transformationen. Collect. Math. 9 (1957), 59–64.

Let P and Q be orientable finite Euclidean manifolds in Euclidean space such that the dimension of Q exceeds one and is one less than the dimension of P. A pair f_1, f_2 of continuous mappings from P into Q is termed normal if the set of points on P where f_1 and f_2 have the same value is either empty or consists of a finite number of pairwise disjoint one-spheres, termed the singularities of (f_1, f_2). For the one-spheres occurring as singularities of a normal pair (f_1, f_2) a concept of equivalence is introduced; the pair (f_1, f_2) is termed minimal if no two of its

singularities are equivalent. For the singularities of a minimal pair (f_1, f_2) a criterion for essentiality is defined. Given a continuous mapping g from P into Q there exists a minimal mapping pair (g_1, g_2) such that g_1 and g_2 are each homotopic to g. The number of essential singularities of (g_1, g_2) is defined to be the multiplicity of g. A proof is sketched for the theorem that this multiplicity is homotopy invariant. *P. V. Reichelderfer* (Columbus, Ohio)

6694:

Eckmann, Beno; et Hilton, Peter J. Groupes d'homotopie et dualité. Groupes absolus. C. R. Acad. Sci. Paris **246** (1958), 2444–2447.

All the main groups of algebraic topology are reduced to the notion of a generalised homotopy group.

Let $\Pi(A, B)$ denote the set of base-point-fixed homotopy classes of maps $F:A\to B$ in the category of spaces with base-point. A space B is said to be an H-space if there is a map $M:B\times B\to B$ having certain well-known properties; a space, dually, provided with a map $M':A\to A\vee A$ is called an H'-space. Let $\Delta:B\to B\times B$ be the diagonal, $\Delta':A\vee A\to A$ the map identifying corresponding points. If B is an H-space or A an H'-space, then $\Pi(A, B)$ has a group structure induced by

$$f\cdot g=M\circ(f\times g)\circ\Delta,$$
$$f+g=\Delta'\circ(f\vee g)\circ M',$$

respectively. Let ΩX be the loop-space on X, ΣX the suspension (with one base point). ΩX has an H-structure, ΣX an H'-structure, and $\Pi(\Sigma A, B)\cong\Pi(A, \Omega B)$. By iteration are defined

$$\Pi_n(A, B)=\Pi(\Sigma^n A, B)\cong\Pi(A, \Omega^n B).$$

The homotopy groups $\pi_n(A)$ of a space are evident special cases. Hence we have the notion of an "Eilenberg-MacLane space" $K(G, m)$ and can define the cohomology groups:

$$H^m(A; G)=\Pi(A, K(G, m))\cong\Pi(A, \Omega^k K(G, m+k)).$$

The homology groups are introduced as $H_m(A; G)=\operatorname{Car} H^m(A, \operatorname{Car} G)$, where Car denotes the character group. By $K'(G, m)$ denote the (Moore-)space which has homology groups $H_i(K')=G$ if $i=m$, 0 otherwise (and $\pi_1=H_1$); then we define $\pi_m(B, G)=\Pi(K'(G, m), B)$ $(m\geqq 2)$, the mth homotopy group of B with coefficients in G.
V. Gugenheim (Baltimore, Md.)

6695:

Eckmann, Beno; et Hilton, Peter J. Groupes d'homotopie et dualité. Suites exactes. C. R. Acad. Sci. Paris **246** (1958), 2555–2558.

The groups $\Pi(A, B)$ and $\Pi_n(A, B)$ [see review above] are generalised to groups $\Pi(\alpha, \beta)$, $\Pi_n(\alpha, \beta)$, where α, β are maps. $\Pi(\alpha, \beta)$ is the set of homotopy classes of commutative diagrams $g\alpha=\beta f$; the operations Σ and Ω apply to maps. Various types of relative groups are introduced with these notions; then the ideas of fibre-space (homotopy lifting) and co-fibre space (homotopy extension) are introduced. Using these notions the authors obtain various exact sequences. *V. Gugenheim* (Baltimore, Md.)

6696:

Eckmann, Beno; et Hilton, Peter J. Groupes d'homotopie et dualité. Coefficients. C. R. Acad. Sci. Paris **246** (1958), 2991–2993.

The exact sequences of the last note [review above] are applied to obtain most of the general exact sequences of algebraic topology; in particular, the authors obtain a coefficient sequence for homotopy groups with coefficients, and deduce a universal coefficient theorem (originally due to Peterson) for these groups.
V. Gugenheim (Baltimore, Md.)

6697:

Eckmann, Beno; et Hilton, Peter J. Transgression homotopique et cohomologique. C. R. Acad. Sci. Paris **247** (1958), 620–623.

Continuation of #6694–6696 above. The idea of the transgression is generalised in the following context. Let $X'\xrightarrow{\xi}X\xrightarrow{\eta}X''$ be a "differential triple", i.e., X', X, X'' are spaces, ξ, η maps such that $\eta\xi$ is trivial; let A be a space. Then using the definitions of the earlier notes the following diagram is defined:

$$\begin{array}{ccccccccc}
\cdots\to\Pi_n(A, X') & \xrightarrow{\xi_*} & \Pi_n(A, X) & \xrightarrow{\mathscr{J}} & \Pi_n(A, \xi) & \xrightarrow{\partial} & \Pi_{n-1}(A, X')\to\cdots \\
\downarrow g_* & (1) & \| & (2) & \downarrow h_* & (3) & \downarrow g_* \\
\cdots\to\Pi_{n+1}(A, \eta) & \xrightarrow{\partial} & \Pi_n(A, X) & \xrightarrow{\eta_*} & \Pi_n(A, X'') & \xrightarrow{\mathscr{J}} & \Pi_n(A, \eta)\to\cdots
\end{array}$$

where g, h are pairs of maps

$$\begin{array}{ccc}
X' & \xrightarrow{g_1=\xi} & X \\
\downarrow & & \downarrow\eta \\
\text{base-point} & \xrightarrow[g_2]{} & X''
\end{array}
\qquad
\begin{array}{ccc}
X' & \xrightarrow{h_1} & \text{base-point} \\
\downarrow\xi & & \downarrow \\
X & \xrightarrow[h_2=\eta]{} & X''
\end{array}$$

in which the horizontal sequences are exact, (1), (2) commute and (3) anticommutes. Then ∂h_*^{-1} and $-g_*^{-1}\mathscr{J}$ define the same homomorphism of $h_*\Pi_n(A, \xi)$ into a quotient-group of $\Pi_{n-1}(A, X')$; it is called the covariant transgression. If the given differential triple is a fibre-map, the vertical homomorphisms in our ladder are isomorphisms, and the diagram becomes the (generalised) exact homotopy sequence.

All of this can be strictly dualised; one is led to a contravariant transgression which contains the classical cohomology-transgression as a special case.

The authors give several applications, already so condensed that further condensation appears impossible.
V. Gugenheim (Baltimore, Md.)

6698:

Puppe, Dieter. Homotopiemengen und ihre induzierten Abbildungen. I. Math. Z. **69** (1958), 299–344.

Let X, V be objects in the category C of spaces with base-points, and let $\pi(X, V)$ denote the set of (based) homotopy classes of maps of X into V. Then $\pi(X, V)$ is a set with zero element, and a map f: $X\to Y$ in C induces a transformation f^*: $\pi(Y, V)\to\pi(X, V)$ respecting zero elements. Let f be used to attach the cone on X to Y and denote the resulting space by C_f. There are then an embedding Pf: $Y\to C_f$ and a projection map Qf: $C_f\to SX$, where S is the suspension functor. The sequence of maps

$$\mathfrak{A}f\colon X\xrightarrow{f}Y\xrightarrow{Pf}C_f\xrightarrow{Qf}SX\xrightarrow{Sf}SY\to\cdots$$

gives rise to a sequence of homotopy sets and transformations

$$(\mathfrak{A}f)^*\colon \pi(X, V)\xleftarrow{f^*}\pi(Y, V)\xleftarrow{Pf^*}\pi(C_f, V)\xleftarrow{Qf^*}\pi(SX, V)$$
$$\xleftarrow{Sf^*}\pi(SY, V)\leftarrow\cdots.$$

This sequence is exact in the sense that the counterimage of the zero element under one of the transformations is the image of the preceding transformation; moreover, to the right of and including $\pi(SX, V)$, the sets are groups and the transformations homomorphic. The sequence $(\mathfrak{A}f)^*$ generalizes the track group sequence of M. G. Barratt [it coincides with the sequence $S^*(\alpha)$ of Eckmann and Hilton,

#6694 above]; the author gives an ingenious proof of exactness by showing that the sequence coincides, up to isomorphism, with that obtained by iterating the process passing from f to Pf. In section 4 the author refines the notion of exactness at $\pi(C_f, V)$ and $\pi(SX, V)$; an operation $\alpha \top a$, $\alpha \in \pi(SX, V)$, $a \in \pi(C_f, V)$, of the group $\pi(SX, V)$ on the set $\pi(C_f, V)$, is defined and used to show that two elements of $\pi(SX, V)$ have the same Qf^*-image if and only if their difference lies in $Sf^*\pi(SY, V)$. Moreover, a, $a' \in \pi(C_f, V)$ have the same Pf^*-image if and only if $a' = \alpha \top a$ for some $\alpha \in \pi(SX, V)$. However, the author shows by examples that the 'cosets' of $\pi(C_f, V)$ under Pf^*, that is, the subsets $Pf^{*-1}(Pf^*a)$, may have different cardinals, in contrast to the situation when group structures are present. Thus, for general maps f, a distinction must be drawn between the statement that f^* is monomorphic ($f^{*-1}(0)=0$) and the statement that f^* is (1, 1). In section 3 the author studies conditions under which f^* is monomorphic, the main result is that, if X and Y are connected polyhedra with Y simply-connected, then f^* is a monomorphism for all V if and only if Sf possesses a right homotopy inverse.

In section 5 the author applies the theory to study the inclusion map $i: X \vee Y \to X \times Y$. Under a suitable assumption on the local properties of the spaces near the base point he proves the formula

$$S(X \times Y) \simeq SX \vee SY \vee S(X \wedge Y),$$

where $X \wedge Y$ is obtained from $X \times Y$ by pinching $X \vee Y$ to a point. This formula generalizes to the product of any finite number of spaces, under assumptions implying that the $\wedge$-operation is associative. *P. J. Hilton* (Ithaca, N.Y.)

6699:

Puppe, Dieter. Homotopiemengen und ihre induzierten Abbildungen. II. Sphärenähnliche Mannigfaltigkeiten. Math. Z. **69** (1958), 395–417.

[For notations, see I, reviewed above.] In this paper the author takes up the question of whether $g\colon X \to Y$ induces a monomorphism $g^*\colon \pi(Y, V) \to \pi(X, V)$ for all V, in the special case when $X = P^n$ is an orientable pseudomanifold, $Y = S^n$, and g is a map of degree 1. In this case g may be identified with a suitable Qf, so that g^* is a monomorphism if and only if it is (1, 1). If P^n has the stated property it is called sphere-like. It is easy to see (using the results of I) that it is sufficient to test g^* when $V = S(P^n - e^n)$, e^n being an n-cell of P^n; and it follows from Satz 12 of I that P^n is sphere-like if and only if there is a map $S^{n+1} \to S(P^n)$ of degree 1, that is, if and only if the fundamental homology class of P^n in dimension n becomes spherical under suspension.

The main results are the following: (1) sums and products of sphere-like pseudomanifolds are again sphere-like; (2) a compact differentiable manifold M^n which is differentiably embeddable in R^{n+1} is sphere-like; (3) a compact differentiable manifold M^n is sphere-like if and only if there exists a compact differentiable submanifold $'M^n$ of R^{n+1} which can be mapped onto M^n with degree 1; (4) the Stiefel-Whitney classes of a compact differentiable sphere-like manifold are trivial; (5) P^3 is sphere-like if and only if the Postnikov squares $H^1(P^3; Z_{2^k}) \to H^3(P^3, Z_{2^{k+1}})$ vanish for all $k \geq 1$. *P. J. Hilton* (Ithaca, N.Y.)

6700:

Milnor, John. On spaces having the homotopy type of CW-complex. Trans. Amer. Math. Soc. **90** (1959), 272–280.

Let W be the class of spaces having the homotopy-type of a C W complex [cf. J. H. C. Whitehead, Bull. Amer. Math. Soc. **55** (1949), 312–245; MR **11**, 48]. The class W has convenient properties for use in homotopy theory, and the present paper is mainly devoted to theorems that allow one to recognise whether a space is in W, and, in particular, what constructions applied to spaces in W yield other spaces in W; more generally, these remarks apply to W^n, the n-ads $(X; X_1, \cdots, X_{n-1})$ which have the homotopy type of a CW n-ad, i.e., X a CW complex, X_i subcomplexes. Theorem: The following restrictions on an n-ad $A = (X; X_1, \cdots, X_{n-1})$ are equivalent: (a) A belongs to W^n; (b) A is dominated by a CW n-ad; (c) A has the homotopy type of a simplicial n-ad in the weak topology; (d) A has the homotopy type of a simplicial n-ad in the strong topology. (A similar theorem characterises spaces having the homotopy type of a countable CW complex.) Theorem: Let A belong to W^n and C be a compact n-ad; then the n-ad of function-spaces A^C belongs to W^n. Theorem: If A belongs to W^n, B to W^m, then $A \times B$ belongs to W^{n+m-1}.
V. Gugenheim (Baltimore, Md.)

6701:

Zisman, Michel. Suite spectrale des fibres au sens de Kan. C. R. Acad. Sci. Paris **248** (1959), 762–764.

The author observes that the method of J. C. Moore and the reviewer [Trans. Amer. Math. Soc. **85** (1957), 265–306; MR **19**, 160] for computing the term E_2 of the spectral sequence of a fibre-space can be applied to Kan-complexes. *V. Gugenheim* (Baltimore, Md.)

6702:

Haefliger, André. Structures feuilletées et cohomologie à valeur dans un faisceau de groupoïdes. Comment. Math. Helv. **32** (1958), 248–329.

Let σ be an analytic foliation of codimension one of a real analytic manifold M. About each point of M there is an analytic coordinate system $x_1, \cdots, x_n$ with domain, say, U such that each component of the intersection of a leaf of σ with U is of the form $\{p \in U \mid x_n(p) = \text{constant}\}$, and the set of such coordinate systems determines σ. A continuous map τ of the unit interval into M is called a transversal of σ if $\tau(t_0) \in U$ implies that $x_n \circ \tau$ is a local homeomorphism of a neighborhood of t_0 into R. The basic lemma of chapter V of this paper states that if τ is a closed transversal of σ, then τ represents an element of infinite order in the fundamental group of M. If M is compact it is easy to construct a C^∞ closed transversal to σ, so it follows from the basic lemma that if M is compact and $\pi_1(M)$ has only elements of finite order, then M admits no analytic foliations of codimension one. This answers negatively a question of Reeb who, having constructed a C^∞ foliation of S^3 of codimension one, asked if there existed analytic ones. The author also applies the basic lemma to investigate the existence, and more generally the number, of compact leaves of σ.

The first four chapters of the paper are concerned with an extreme, Bourbaki-like generalization of the notion of foliation. After some twenty-five pages and several hundred preliminary definitions, the reader finds that a foliation of X is to be an element of the zeroth cohomology space of X with coefficients in a certain sheaf of groupoids. Holonomy, the Reeb-Ehresmann stability theorems, etc., are then generalized to this setting. While such generalization has its place and may in fact prove useful in the future, it seems unfortunate to the reviewer that the author has so materially reduced the accessibility of the results, mentioned above, of Chapter V,

by couching them in a ponderous formalism that will undoubtedly discourage many otherwise interested readers. *R. Palais* (Princeton, N.J.)

6703:

Chamberlin, R. E. A class of unknotted curves in 3-space. Proc. Amer. Math. Soc. **10** (1959), 149–157.

Let M_n be a 2-dimensional manifold in E_3 which is a sphere with n handles, $f_i(C)$ a set of pairwise disjoint images of a circle each lying on M_n and such that $f_i(C)$ is homologous to zero. Then there are pairwise disjoint images of a disk $g_i(D)$ in E_3 such that $f_i(C)$ is the boundary of $g_i(D)$ and $g_i|D-C$ is interior to M_n.

C. B. Allendoerfer (Seattle, Wash.)

6704:

Yang, C. T. Transformation groups on a homological manifold. Trans. Amer. Math. Soc. **87** (1958), 261–283.

Let $\mathfrak{C}$ be a compact abelian coefficient group. A homological n-manifold of type $(\mathfrak{C}, n)$ is a connected, finite-dimensional, locally compact Hausdorff space with a local restriction denoted by $P_n(\mathfrak{C})$. Except that local compactness replaces compactness and $\mathfrak{C}$ replaces a field, this definition, as well as the description of P_n, is due to P. A. Smith, who established some of the key properties. The present article restates many of these properties in more modern terms and with more attention to $\mathfrak{C}$ or $\mathfrak{P}$ (the reals mod 1). It is shown that a property $Q(\mathfrak{C})$ (also used by Smith) actually follows from $P_n(\mathfrak{C})$ for locally compact Hausdorff spaces. The author extends from the ordinary n-manifold case to the $(\mathfrak{C}, n)$ manifold, X, the known result that if G is a compact Lie group on X with orbit dimension at most r, then the union of orbits of dimension $\leqq k$, $k<r$, is a closed set of dimension $\leqq n-r+k-1$. *D. G. Bourgin* (Urbana, Ill.)

6705:

Montgomery, D.; and Yang, C. T. Orbits of highest dimension. Trans. Amer. Math. Soc. **87** (1958), 284–293.

Let X be a $(\mathfrak{P}, n)$ manifold [#6704 above]. Let G be a compact, connected Lie group acting on X. The main result, known for the special case that X is a differentiable n-manifold and that G acts differentiably, is that if $H_{n-1}(X, I_2)=0$, then $\dim E<n-1$. Here E is a subset of the set of r-dimensional orbits characterized loosely by the fact that the stability group G_x at $x \in E$ is discontinuous, yet G_x does not have a connected, invariant slice cutting x. *D. G. Bourgin* (Urbana, Ill.)

DIFFERENTIAL GEOMETRY, MANIFOLDS

See also 6481, 6537, 6538.

6706:

Maitre, Jean. Sur les congruences "W" de Weingarten. Enseignement Math. (2) **4** (1958), 108–119.

L'auteur se propose d'exposer sous forme condensée certains résultats classiques qui concernent les congruences "W". {Signalons à ce propos que la même lettre D est employée pour désigner à la fois le rayon de la congruence et le premier coefficient de la deuxième forme fondamentale. Quelques erreurs typographiques s'y sont également glissées. À la page 111, au second membre de l'égalité vectorielle qui donne P_{22}, le coefficient du vecteur N_2 doit être $(\lambda'+2\theta_2)$. À la page 114, la valeur du vecteur P_2 doit être: $P_2=R_2N-\theta N_2$. Enfin toujours à la même page, λ' n'est pas égal à R_2, mais à R_2'.}

F. Şemin (Istanbul)

6707:

Mineo, Corradino. Sulle rappresentazioni isodromiche di una superficie sopra un'altra. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **24** (1958), 227–230.

A representation between two surfaces is said to be isodromic [A. Venturi, 1898] if all isogonal trajectories to a given ∞^1 system of curves on one surface are transformed in isogonal trajectories of a system of ∞^1 curves of the other surface (angles may not be preserved).

The author proves that isodromic transformations depend on an arbitrary harmonic function.

E. Bompiani (Rome)

6708:

Blaschke, Wilhelm. Sulle geodetiche chiuse. Boll. Un. Mat. Ital. (3) **13** (1958), 240–247.

Esposizione chiara, elegante dei principali risultati relativi al problema delle geodetiche chiuse dovuti principalmente ad Artin, Caratheodory, Darboux, Funk, Klingenberg, Pogorelof, Poincaré. Sono accennati anche metodi per riottenere alcuni risultati e segnalati problemi ancora non risoluti. *C. Longo* (Parma)

6709:

Rimini, Cesare. Contributo alla impostazione del calcolo tensoriale. Accad. Sci. Modena. Atti Mem. (5) **15** (1957), 142–176.

An elementary exposition of tensor calculus with an invariant quadratic differential form.

E. Bompiani (Rome)

6710:

Rzewuski, J. Two theorems concerning the field equations in the spinor space. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 335–341.

6711:

Trupin, Š. Collinearity and coplanarity of affinors in a linear dimensionless space. Latvijas PSR Zinātņu Akad. Vēstis **1958**, no. 8(133), 83–92. (Russian)

On dit que les affineurs $Ax_1\cdots x_k$, $By_1\cdots y_l$, $Cz_1\cdots z_m$ sont coplanaires avec les vecteurs (linéairement indépendants) $p_1, \cdots, p_n$ si

$$[\overset{k}{A}x\cdots x\overset{l}{B}x\cdots x\overset{m}{C}x\cdots xp_1\cdots p_n]=0$$

pour chaque x. L'A. prouve le théorème: Si les affineurs Ax_1x_2, By, Cz sont coplanaires avec $p_1, \cdots, p_n$, alors des fonctions linéaires scalaires $\underline{A}xy$, $Bxyz$, Γxyz, $\Delta_i xyzw$ $(i=1, \cdots, n)$ existent de sorte que l'égalité

$$\underline{A}xx\cdot Axx+\underline{B}xxx\cdot Bx+\Gamma xxx\cdot Cx+\sum_{i=1}^{n} p_i\Delta_i xxxx=0$$

soit valable identiquement en x. *A. Švec* (Prague)

6712:

Švelkin, P. I. Invariant constructions on an m-dimensional surface in n-dimensional affine space. Dokl. Akad. Nauk SSSR **121** (1958), 811–814. (Russian)

Une variété $V_m \subset A_n$ soit donnée par l'équation

$$d\vec{\Lambda}=\omega^a\Lambda_a{}^\alpha E_\alpha \quad (\alpha=1, \cdots, n;\ a=1, \cdots, m)$$

où ω^a sont les formes de Pfaff. Le prolongement de cette équation conduit à l'objet fondamental $\{\Lambda\}_w$ d'ordre w; le champ de cet objet d'ordre suffisamment grand peut embrasser le champ engendré par l'objet quelconque [v.

Laptev, Trudy Moskov. Mat. Obšč. **2** (1953), 275–382; MR **15**, 254]. On construit quelques tenseurs et scalaires relatifs et on étudie les embrassements des objets normaux qui correspondent aux objets des connections de Hlavatý [Nederl. Akad. Wetensch. Proc. **52** (1949), 505–517, 714–724, 977–986; MR **11**, 54; **12**, 232]. On construit une normalisation invariante et un repère canonique de la variété envisagé. On trouve un système des invariants de sorte que chaque invariant qui peut être embrassé par l'objet fondamental est leur fonction. Toutes les affirmations sont citées sans démonstrations.

A. Švec (Prague)

6713:

Santaló, L. A. Affine differential geometry and convex bodies. Math. Notae **16** (1957), 20–42. (Spanish)

The paper contains the fundamental formulae of the affine differential geometry of surfaces obtained by the method of moving frames of E. Cartan. As parametric curves the affine lines of curvature are used, which are the most appropriate for the study of convex surfaces. The method furnishes simply and naturally certain integral formulae which are used to prove some inequalities between the volume V, affine area Ω and integrated affine mean curvature M of a closed convex surface (for instance, $\Omega^2 \leq 3MV$, equality only for the ellipsoids) and some characterizations of the ellipsoids due to Blaschke [Vorlesungen über Differentialgeometrie, II, Springer, Berlin, 1923] and Grotemeyer [Arch. Math. **3** (1952), 38–43, 307–310; **4** (1953), 230–233; MR **14**, 788, 789; **15**, 341].

Summary provided by the author

6714:

Italiani, Mario. Sulle congruenze di piani dello spazio proiettivo S_4. Boll. Un. Mat. Ital. (3) **13** (1958), 105–111.

È una nota di carattere riassuntivo in cui si raccolgono i risultati relativi alla classificazione proiettiva differenziale dei sistemi analitici (ma forse basta differenziabili) ∞^2 (congruenze) di piani dello spazio proiettivo complesso a quattro dimensioni. Per le dimostrazioni si rimanda ad una memoria di prossima pubblicazione negli Atti Accad. Sci. Ist. Bologna.

F. Gherardelli (Florence)

6715:

Fedenko, A. S. Symmetric spaces with simple noncompact fundamental groups. Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 1026–1028. (Russian)

On the basis of results of E. Cartan [*Oeuvres complètes, I, Groupes de Lie*, Gauthier-Villars, Paris, 1952; MR **14**, 343] and F. I. Karpelevič [Trudy Moskov. Mat. Obšč. **4** (1955), 3–112; MR **19**, 384] the author undertakes a classification of symmetric spaces with simple noncompact fundamental groups. The article enumerates models of symmetric spaces of this class for simple noncompact groups of the basic series: ${}^lA_n$, $\tilde{D}_l$, ${}^{2l}C_n$. In conjunction with existing results [B. A. Rozenfel'd, *Non-Euclidean geometries*, Moscow, 1955; MR **17**, 293], this gives a complete solution of the question of classifying those symmetric spaces for which the fundamental groups are real forms of simple classical groups (for brevity we shall denote this class of spaces by L^c). This classification leads the author to the following results: a) all L^c spaces are irreducible Riemann spaces, b) the L^c spaces fall into two subclasses, one containing spaces with metric null-signature, the other all the rest. The latter are arranged in pairs, with common stationary subgroup but distinct fundamental groups belonging to one complex structure.

In conclusion, an enumeration is given of stratifiable L^c spaces.

N. S. Sinyukov (RŽMat **1957** #8200)

6716:

Pa, Chen-kuo. On the equations of structure of a Riemannian space. Sci. Record (N.S.) **1** (1957), 199–203.

Cartan's structure equations for a Riemannian V_n are interpreted for a net of curves belonging to a surface imbedded in a $V_3 \subset V_n$.

Some of the equations are intrinsic to the surface, the others to its imbedding in V_3. Particular consequences for special families of surfaces in V_3.

E. Bompiani (Rome)

6717:

Šapiro, Ya. Geodesic direction fields and groups of similitudes in spaces. Dokl. Akad. Nauk SSSR **120** (1958), 481–484. (Russian)

Soient $\overset{a}{A}{}^\alpha$ ($a=1, \cdots, m>1$) les vecteurs qui engendrent les champs géodésiques des directions dans l'espace à connexion affine A_{n+m} [v. Šapiro, mêmes Dokl. **32** (1941), 237–239; Mat. Sb. N.S. **36(78)** (1955), 125–148; MR **3**, 191; **17**, 79] et supposons la normalisation des $\overset{a}{A}{}^\alpha$ faite de manière que (1) $A \equiv \sum \lambda_a \overset{a}{A}{}^\alpha$ (pour λ_a=const. quelconque) engendre un champ géodésique; (2) $A \equiv 0$, λ_a=const. implique $\lambda_a=0$; alors l'ensemble des champs engendrés par tous les vecteurs A s'appelle variété linéaire (à $m-1$ dimensions) des champs géodésiques des directions. L'A. trouve les conditions pour les $\overset{a}{A}{}^\alpha$ qui forment une telle variété et il donne la forme canonique des composantes de la connexion d'A_{n+m} qui possède m champs géodésiques ou une variété des champs géod. à $m-1$ dimensions.

Enfin on énonce (également sans démonstrations) quelques résultats sur les groupes des automorphismes homothétiques d'A_{m+n} définis comme suit: Si $\overset{a}{A}{}^\alpha(\partial/\partial x^\alpha)$ sont les opérateurs du groupe envisagé, alors le champ des directions engendré par le vecteur $\sum \lambda_a \overset{a}{A}{}^\alpha$ d'un sous-groupe quelconque est géodésique.

A. Švec (Prague)

6718:

Haimovici, Adolf. Sur quelques invariants au transport parallèle, dans les espaces à connexion affine. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Mat. **8** (1957), no. 2, 135–149. (Romanian. Russian and French summaries)

The following problem is treated in the present paper: To determine functions of the components of a tensor in a space with affine connection X^n which are invariant under the parallel transport of the tensor. This is a generalization of a problem treated by the author in previous papers [see Rend. Mat. e Appl. (5) **15** (1956), 385–452; MR **20** #3579].

The problem reduces to a system of n partial differential equations of the first order and the Poisson brackets obtained from it. This system is linear with coefficients which are linear in the components of the tensor and can be studied in the same way as in the paper mentioned above. Some of the results obtained are:

If X^n admits r invariants attached to a tensor of order m then it admits $r'>r$ invariants attached to a tensor of order $m'>m$.

Every X^n admits at least $n^2(n-1)$ invariants attached to a tensor of order 3. These invariants are found explicitly in the case $n=2$.

Every X^2 admits one invariant attached to a co- or

contravariant tensor of order two, and two invariants attached to a mixed tensor of order two. All these invariants are given explicitly, together with their geometric interpretation.

The paper concludes with a study of those X^n which admit 1) the invariant function $\varphi(x^i)\det\|g_{rs}\|$; 2) the invariant functions $\varphi(x^i)g_{rs}$. *R. Blum* (Saskatoon, Sask.)

6719:

Mastrogiacomo, Pasquale. Sulle connessioni tensoriali per tensori controvarianti e convarianti m-pli. Giorn. Mat. Battaglini (5) 5 (85) (1957), 305–321.

Tensor connections (as different from the usual vector connections) have been introduced by E. Bompiani [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1 (1946), 478–482; MR 8, 404] and largely studied by A. Cossu.

They are defined by a system of components $L_{\cdot\cdot\cdot}^{\cdot\cdot\cdot}$ satisfying a fundamental system of partial differential equations allowing the definition of covariant (tensor) derivatives and differentials. By symmetrization and alternation two tensors Φ, Ψ can be deduced from the L's. Their vanishing is the necessary and sufficient condition for the permanence of the symmetric and alternating character of a contravariant or covariant tensor through differentiation. Relations between these two tensors are examined (they coincide if the connection is defined for double tensors, the case studied by Bompiani). Tensor connections giving rise to the same tensors Φ and Ψ are determined.

The geometric meanings of Φ and Ψ and of $\Phi\equiv\Psi$ are also given. Tensor connections of particular types (for instance, deduced from one or more vector connections) are examined. *E. Bompiani* (Rome)

6720:

Allamigeon, André-Claude. Isomorphisme des connections infinitésimales. C. R. Acad. Sci. Paris **246** (1958), 220–222.

Soient $H(V_m, G, p)$ et $\bar{H}(V_m, \bar{G}, \bar{p})$ deux espaces fibrés principaux à groupe structural de Lie, de même base V_m, et soit φ un isomorphisme de H sur $\bar{H}$ tel que: $\bar{p}\circ\varphi=p$, $\varphi(hg)=\varphi(h)\sigma(g)$ $(h\in H,\ g\in G)$, où σ est un isomorphisme différentiable de G sur $\bar{G}$. L'isomorphisme φ fait correspondre à toute connexion infinitésimale [C. Ehresmann, Colloque Topologie, Bruxelles, 1950, pp. 29–55; Thone, Liège, Masson, Paris, 1951; MR 13, 159] $\mathscr{H}$ sur H, une connexion $\bar{\mathscr{H}}$ sur $\bar{H}$. On établit des relations entre les formes de connexion et courbure de $\mathscr{H}$ et celles correspondantes de $\bar{\mathscr{H}}$.

Lorsque $H=\bar{H}$=espace des repères linéaires de V_m, les cas particuliers suivants sont étudiés: (a) σ est l'application identique de $G=\mathrm{GL}(m)$; (b) σ transforme chaque matrice en sa contragrédiente; dans le cas des connexions $L_{\beta\gamma}{}^{\alpha}$ de la théorie unitaire d'Einstein, on retrouve les résultats de Mme Maurer [C. R. Acad. Sci. Paris **246** (1958), 240–243; MR 19, 1141].

Le cas des espaces homogènes est envisagé. Si $\mathscr{H}$ est une connexion de Cartan au sens d'Ehresmann, il en est de même de $\bar{\mathscr{H}}$.

L'étude précédente permet d'étendre les résultats de Mme Cattaneo [Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. **22** (1957), 146–154; MR 19, 1076] sur les connexions affines et ceux de Mme Maurer [loc. cit.] sur les connexions "coaffines". *P. Libermann* (Rennes)

6721:

✱Chern, Shiing-shen. Geometry of submanifolds in a complex projective space. Symposium internacional de topología algebraica [International symposium on algebraic topology], pp. 87–96. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958. xiv+334 pp.

Verf. zeigt in diesem Vortrag, welche Rolle der Krümmungsbegriff in der algebraischen Geometrie und verwandten Disziplinen spielt. In Zusammenhang mit charakteristischen Klassen hat die Krümmungstheorie auch der algebraischen Topologie Anregungen gegeben. Man denke etwa an den Satz von A. Weil [Chern, Topics in differential geometry, Institute for Advanced Study, Princeton, N.J., 1951; MR 19, 764; pp. 58–59]. Alle Invarianten einer differenzierbaren bzw. algebraischen Mannigfaltigkeit, die durch charakteristische Zahlen ausgedrückt werden können (Index, arithmetisches Geschlecht) lassen sich als Integrale über aus dem Krümmungstensor abgeleitete Formen darstellen. Verf. diskutiert folgende Aspekte der Theorie.

1. (Local geometry of submanifolds.) Oskulierende Räume höherer Ordnung.

2. (Integral-geometric invariants.) Volumen einer Untermannigfaltigkeit des $P_N(C)$ und der ihr zugeordneten Räume. Die Ordnung vom Rang p einer im $P_N(C)$ eingebetteten algebraischen Kurve: Interpretation durch das von den oskulierenden Räumen der Ordnung p überstrichene Volumen und nach Santaló [Amer. J. Math. **74** (1952), 423–434; MR 13, 971]. Darstellung der höheren Ordnungen durch Integrale ermöglicht Definition auch für nicht-kompakte Kurven, was in der Theorie von Weyl-Ahlfors [Ahlfors, Acta Soc. Sci. Fenn. Nova Ser. A 3 (1941), no. 4; MR 2, 357; und H. Weyl, *Meromorphic functions and analytic curves*, Princeton Univ. Press, 1943; MR 5, 94] von entscheidender Bedeutung ist. — Analoge Begriffsbildungen für Untermannigfaltigkeiten Euklidischer Räume führen zu andersgearteten Resultaten [Chern, Enseignement Math. **40** (1951–1954), 26–46 (1955); MR 16, 856; und Chern and Lashof, Amer. J. Math. **79** (1957), 306–318; MR 18, 927].

3. (Exterior differential forms on a submanifold.) Plückersche Formeln. Arbeiten von H. Weyl, J. Weyl und Ahlfors. Abschätzung nach oben für das Geschlecht einer algebraischen Kurve der Ordnung d, die im $P_N(C)$ aber nicht im $P_{N-1}(C)$ liegt [Chern, Abh. Math. Sem. Univ. Hamburg **11** (1936), 163–170]. Geometrische Sätze, die mit Hilfe der Tatsache bewiesen werden, dass die Differentiale gewisser geometrischer Grössen meromorphe Differentialformen sind. Indentität von Liebmann [S.-B. Math. Nat. Kl. Bayer. Akad. Wiss. **1927**, 73–87].

"Finally, it may be remarked that explicit exhibition of exterior differential forms on a submanifold in the real Euclidean space has also numerous geometrical consequences... Many classical theorems in differential geometry in the large can be most quickly proved by picking a suitable differential form and using the fact that the integral of its exterior derivative over the manifold is zero. Among these theorems are: the 'Unverbiegbarkeit' of the sphere, Cohn-Vossen's rigidity theorem as proved by Herglotz, the Christoffel-Hurwitz and the Minkowski uniqueness theorems, etc."

F. Hirzebruch (Bonn)

6722:

Takasu, Tsurusaburo. Global differential geometries of principal fibre bundles in the forms of almost Kleinean geometries by non-connection methods. I. Yokohama Math. J. 6 (1958), 1–77.

"The most important problem of geometry seems to be a generalization of the Erlanger Programm of Felix Klein (1872) to the case of differentiable manifolds in the large. . . The purpose of the present paper consists in solving this main problem, offering a series of global differential geometries of principal fibre bundles in the forms of almost Kleinean geometries by non-connection methods." *From the author's summary*

6723:

Boyarskiĭ, B. V.; and Vekua, I. N. Proof of the rigidity of piece-wise regular closed convex surfaces of non-negative curvature. Izv. Akad. Nauk SSSR Ser. Mat. 22 (1958), 165–176. (Russian)

On établit la rigidité d'une surface fermée convexe qui est collée d'un nombre fini de surfaces régulières à courbure de Gauss nonnégative. Ce résultat est prouvé à l'aide d'une identité intégrale dont s'est servi M. Blaschke pour démontrer la rigidité des ovaloïdes.

A. Švec (Prague)

PROBABILITY

See also 6664, 6953, 6955.

6724:

Varadarajan, V. S. On a problem in measure-spaces. Ann. Math. Statist. 29 (1958), 1275–1278.

Es sei $\mathscr{B}$ das System der Borelschen Mengen eines separablen vollständigen metrischen Raumes X, μ ein festes abstraktes Wahrscheinlichkeitsmaß und $\mathscr{F}$ das System der Wahrscheinlichkeitsmaße P auf $\mathscr{B}$, die vermöge $P(A)=\mu(\xi^{-1}(A))$ durch eine zufällige Variable ξ mit Werten in X induziert werden. Dann ist $\mathscr{F}$ abgeschlossen hinsichtlich der schwachen Konvergenz von abzählbaren Folgen gegen Wahrscheinlichkeitsmaße. Ein Beispiel zeigt, daß die Voraussetzung, X sei vollständig, nicht entbehrt werden kann. *K. Krickeberg* (Heidelberg)

6725:

Weibull, Christer. The distribution of reciprocal choices in sociometric tests. Stat. Inst. Univ. Gothenburg Publ. **1958**, no. 4, 16 pp.

The main problem considered is the following: in a group of N individuals, each individual chooses a others of the group; if the $\binom{N-1}{a}$ possibilities are equiprobable, what is the probability distribution of the number of reciprocal choices? The principle of inclusion and exclusion is used to give, effectively, the first five factorial moments, and it is shown that the limiting distribution for N increasing is Poisson with means $a^2/2$. Then this is extended to the case where the number of choices varies from individual to individual, the results including only the mean and variance. Finally some remarks are made on triangular and quadrangular reciprocal choices.

J. Riordan (New York, N.Y.)

6726:

Georgiu, Šerban. Some problems on the division of an interval of the straight line by sample points. Ž. Čist. Prikl. Mat. 1 (1956), 109–139. (Russian)

The author finds the characteristic function of an arbitrary linear combination of the lengths of the random intervals into which the interval [0, 1] is divided by n points distributed uniformly and independently on this interval. *J. L. Doob* (Urbana, Ill.)

6727:

Braumann, Pedro. Some remarks about infinitely divisible probability laws. Univ. Lisboa Revista Fac. Ci. A (2) 6 (1957/58), 265–268.

Simple properties of infinitely divisible laws, immediate from P. Lévy's formula.

H. P. McKean, Jr. (Cambridge, Mass.)

6728:

Mott, J. L. The central limit theorem for a convergent non-homogeneous finite Markov chain. Proc. Roy. Soc. Edinburgh Sect. A 65 (1959), 109–120.

"The distribution of x_n, the number of occurrences of a given one of k possible states of a non-homogeneous Markov chain $\{P_j\}$ in n successive trials, is considered. It is shown that if $P_n\to P$, a positive-regular stochastic matrix, as $n\to\infty$, then the distribution about its mean of $x_n/n^{\frac{1}{2}}$ tends to normality, and that the variance tends to that of the corresponding distribution associated with the homogeneous chain $\{P\}$". (From the author's summary)

The method used is to show that the moments converge to the moments of the limiting distribution.

J. Wolfowitz (Ithaca, N.Y.)

6729:

Varadarayan, V. An existence theorem for probability spaces. Uspehi Mat. Nauk 13 (1958), no. 5(83), 167–170. (Russian)

It is proved that there is a countable family of random variables, with arbitrarily preassigned compatible joint distributions, on any probability measure space whose measure is non-atomic and separable. A simple counter-example shows that the theorem is false for non-countable families. *J. L. Doob* (Urbana, Ill.)

6730:

★**Blackwell, David. The entropy of functions of finite-state Markov chains.** Transactions of the first Prague conference on information theory, statistical decision functions, random processes held at Liblice near Prague from November 28 to 30, 1956, pp. 13–20. Publishing House of the Czechoslovak Academy of Sciences, Prague, 1957. 354 pp. Kčs 34.00.

Let $\{x_n, n=0, \pm 1, \cdots\}$ be a stationary ergodic finite-state Markov process with states $i=1, \cdots, I$ and transition matrix $\|m(i, j)\|$. Let Φ be a function defined on $1, \cdots, I$ with values $a=1, \cdots, A\leqq I$, and let $y_n=\Phi(x_n)$. Then $\{y_n\}$ is an ergodic stationary process, the general formula for the entropy of such processes being $H=-E\log_2 P(y_1|y_0, y_{-1}, \cdots)$. Let $\alpha_n=(\alpha_{n1}, \cdots, \alpha_{nI})$, where $\alpha_{ni}=P(x_n=i|y_n, y_{n-1}, \cdots)$. It is shown that $\{\alpha_n\}$ is a stationary Markov process with stationary distribution Q, where Q is a distribution on vectors $(w_1, \cdots, w_I)$, $w_i\geqq 0$, $\sum w_i=1$, satisfying (*) $Q(E)=\sum_a \int_{f_a(w)\in E} r_a(w)dQ(w)$, where $r_a(w)=\sum_i \sum_{\Phi(j)=a} w_i m(i, j)$, and f_a is a vector function of w whose jth component is equal to 0 if $\Phi(j)\neq a$, and equal to $\sum w_i m(i, j)/r_a(w)$ if $\Phi(j)=a$. Then the entropy of $\{y_n\}$ is given by $H=-\int \sum_a r_a(w)\log_2 r_a(w)dQ(w)$. Under additional conditions it is shown that Q is the only probability distribution that is a solution of (*) and that if Q is continuous it is in a certain sense singular.

T. E. Harris (Santa Monica, Calif.)

6731:

Fuchs, A. Un problème de temps d'atteinte. Publ. Inst. Statist. Univ. Paris 7 (1958), 161–166.

A time-homogeneous discrete Markov chain with a finite number of states is regular and has an absorbing state, j. It is shown that the mean time to absorption in this state starting from state i is $1-\sum_{n=1}^{\infty}[P_{ij}{}^n-\pi_j]$, where $P_{ij}{}^n$ is the usual probability of a transition from state i to state j in n steps, and $\pi_j=\lim_{n\to\infty}P_{ij}{}^n$.

D. V. Lindley (Cambridge, England)

6732:

Bertotti, Bruno. Trasformazioni di coordinate e movimento browniano. Boll. Un. Mat. Ital. (3) 13 (1958), 217–223.

Let V_n be an n-dimensional differentiable manifold and let $P(y, s; X, t)$ be the probability for a "particle" which is at $y\in V_n$ at the time s to be in $X\subset V_n$ at the time $t>s$. Given a transformation of coordinates on V_n of the form $x^i=f^i(x'^j)$ $(i, j=1, \cdots, n)$, where the functions f^i are analytic, the author formally expands each of the moments

$$\int_{V_n}(x^i-y^i)P(y,s;d^nx,t), \int_{V_n}(x^i-y^i)(x^k-y^k)P(y,s;d^nx,t), \cdots$$

in a series of linear combinations of moments with respect to the accented variables by integrating term by term the MacLaurin expansions of the integrands about y'. Then he states a few properties of moments which are, by virtue of this expansion, easily seen to be invariant under the above transformation of coordinates.

H. M. Schaerf (St. Louis, Mo.)

6733:

Borovkov, A. A. Some problems concerned with large deviations of the maximum of sums of independent equally distributed random variables. Dokl. Akad. Nauk SSSR 121 (1958), 13–15. (Russian)

Let $x_1, x_2, \cdots$ be a sequence of independent identically distributed random variables with a discrete distribution given by (1) $P(x=m_k)=p_k$, $1\leq k\leq r$, $\sum_{k=1}^{r}p_k=1$, m_k integers, $m_1<0$, $m_r>0$. Let $s_n=\sum_{j=1}^{n}x_j$, $S_n=\max_{1\leq i\leq n}s_i$. The author is interested in the distribution of S_n for large n. He formulates his results in terms of the probability $u_{x,n}$ of the first passage occurring at time n for a random walk with absorbing barrier at 0 and particle in the initial position $x>0$ at time 0. The author obtains the asymptotic development of $u_{x,n}$ for $n\to\infty$ in the following cases: x independent of n; $x=x(n)=o(n^{\frac{1}{2}})$; $n^{\frac{1}{2}}/x(n)=O(1)$, $x(n)=o(n)$. The results were too complicated for reproduction here. Some remarks were made concerning results for more general distributions than (1) and also concerning a random walk with two absorbing barriers. No proofs were given.

L. Schmetterer (Berkeley, Calif.)

6734:

Carleson, Lennart. A mathematical model for highway traffic. Nordisk Mat. Tidskr. 5 (1957), 176–180, 213. (Swedish. English summary)

The mathematical model in question is similar to the models proposed by the reviewer [J. Operations Res. Soc. Amer. 3 (1955), 176–186; MR 17, 985] and by M. S. Bartlett [#6735]. The traffic density is sufficiently low that passing involves an interaction only between pairs of cars, and the various models differ mainly in the assumption regarding the amount of delay experienced by a fast car that wishes to pass a slower car. Here it is assumed that if a car with "ideal speed" x wishes to pass one of ideal speed y, it travels in the outer lane a distance $\lambda=cx/(x-y)$, with c a constant, and that the fast car travels with the slower velocity y for a distance $s(\lambda)$ before passing. Most of the mathematical details deal with the special case in which $s(\lambda)$ is a linear function of λ. The author computes, among other things, the average velocity of a car with ideal speed x and shows that it is bounded even for $x\to\infty$.

G. Newell (Stockholm)

6735:

Bartlett, M. S. Some problems associated with random velocity. Publ. Inst. Statist. Univ. Paris 6 (1957), 261–270.

In § 1 one dimensional processes in which the velocity is Markovian are discussed using operational methods. In particular, it is shewn that Goldstein's hyperbolic differential equation [Quart. J. Mech. Appl. Math. 4 (1951), 129–156; MR 13, 960; see also Scheidegger, Canad. J. Phys. 36 (1958), 649–658; MR 19, 1090] can only arise under the circumstances described by him: namely when only two possible values for the velocity are allowed. In § 2 the same processes are considered with absorbing and reflecting barriers. Integral equations for the joint probability densities of position and velocity are derived but not solved. The remainder of the paper is devoted to a stochastic study of the Lighthill-Whitham [Proc. Roy. Soc. London. Ser. A 229 (1955), 317–345; MR 17, 310] theory of road traffic flow. An equation for the joint density of position and velocity is derived. The problem of estimation of the empirical relation between the space and time densities is discussed and sampling errors of some estimates obtained. Finally two models of how overtaking takes place in a single stream are discussed and theoretical expressions for the same relation obtained.

D. V. Lindley (Cambridge, England)

6736:

Jackson, James R. Multiple servers with limited waiting space. Naval Res. Logist. Quart. 5 (1958), 315–321.

The queueing system $M/M/s$ is studied with the modification that customers arriving and finding $s+W$ customers in the system are "subcontracted" out of the system (i.e., play no further role in its development). Let i be the long-term-average proportion of customers who are "subcontracted". Tables and graphs are given that enable W to be found as a function of i, s and the traffic intensity. [See also P. D. Finch, J. Roy. Statist. Soc. Ser. B. 20 (1958), 182–186; MR 20 #3612.]

D. V. Lindley (Cambridge, England)

6737:

Haight, Frank A. Two queues in parallel. Biometrika 45 (1958), 401–410.

The traffic system considered is that of Poisson arrivals to two servers, each with exponential distribution of service time, but not necessarily with the same service rate, and with delays permitted. Delayed arrivals form two queues, joining the shorter, or if the two are equal, a particular one called the "near" one, and are served from each in order of arrival; defections are not considered. Stationary state probabilities and their moments are determined, in principle and explicitly, respectively, for the two cases (i) no changing of queues permitted and (ii) changes tending to equalize queues permitted. The latter results in a considerable simplification. The procedure is by formulation of the usual differential recurrence relations for state changes in time, but in the first case the rule for queue formation requires careful consideration of arrival effects.

J. Riordan (New York, N.Y.)

6738:

Kendall, David G. La propagation d'une épidémie ou d'un bruit dans une population limitée. Publ. Inst. Statist. Univ. Paris **6** (1957), 307–311.

A population of N susceptibles has introduced into it at time $t=0$ a single infected individual. If s_t is the number of secondary infections at time $t=0$ the probability of a new infection in $(t, t+dt)$ is $\beta(N-s_t)(s_t+1)dt+o(dt)$. The time scale is then chosen so that $\beta=1$. T, the time taken before all the population has received the infection, is the sum of N independent random variables I_j, the time interval during which $s_t=j$, each of which has a negative exponential distribution with parameter $[j(N-j+1)]^{-1}$. The characteristic function of T can thus immediately be written down. From this it is shewn that the limiting probability law as $N\to\infty$ of $w=(N+1)T-2\log N$ is $2K_0(2e^{-\frac{1}{2}w})e^{-w}dw$. The basis of this demonstration is the fact that $T=S_1+S_2$, where S_1 is the time taken for half the population to be infected. S_1 and S_2 are independently and identically distributed according to an extreme value distribution. *D. V. Lindley* (Cambridge, England)

6739:

Koronkevič, O. I. Ergodic properties of random functions in the solution of a linear dynamic system. Dopovidi Akad. Nauk Ukrain. RSR **1958**, 810–812. (Ukrainian. Russian and English summaries)

The author discusses the connection between the conditions satisfying the ergodic theorem for a given multidimensional random function $\xi(t)$ and the conditions satisfying the ergodic theorem for a special solution of the system

$$\frac{dY}{dt}=P(t)Y+\xi(t).$$

Author's summary

6740:

Takács, Lajos. A probability method for the treatment of the secondary electron emission. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **6** (1956), 199–211. (Hungarian)

The probability distribution p_j ($j=0, 1, 2, \cdots$) of obtaining j secondary electrons from a target for one impinging primary electron can be determined if the output amplitude distribution of an n-stage electron multiplier is measured. If the latter is characterized by the probability distribution $P\{\nu_n=j\}=P_j$ ($j=0, 1, 2, \cdots$), where the variable ν_i ($i=1, 2, \cdots$) denotes the number of secondaries in the ith stage, then the mathematical problem is to calculate p_j from P_j. The author presents the solution for the case that the dynode structures are equal, meaning that p_j is the same for all stages. First on the basis of the theory of cascade processes the generator functions $u_i(t)$ are calculated from the generator function $u(t)=\sum_{j=0}^{\infty} p_j t^j$ of p_j, and then the binomial moments of ν_1 are determined from the known binomial moments of ν_n. Some approximate and practical methods for the calculations are given in the paper. *Zoltan Bay* (Washington, D.C.)

STATISTICS

See also 6799.

6741:

Boyd, A. V. Bounds for Mills' ratio for the type III population. Ann. Math. Statist. **29** (1958), 926–929.

Monotone sequences of upper and lower bounds are found for the ratio of the density function at x of a standardized type III Pearson distribution to the area under the curve to the right of x. The estimates of the ratio are gotten using a continued fraction expansion of the incomplete gamma function. *S. W. Nash* (Vancouver, B.C.)

6742:

Birnbaum, Z. W. On an inequality due to S. Gatti. Metron **19** (1958), no. 1–2, 243–244.

Let $a_1, a_2, \cdots, a_n$ be real numbers and let $\bar{a}=\sum_{i=1}^n a_i/n$. Then, for any $r\geqq 1$,

$$n^{-1}\sum_{i=1}^n |a_i-\bar{a}|^r\leq\left(\frac{n}{2}\right)^{r-1}\left(\sum_{i=1}^n |a_i-\bar{a}|\right)^r.$$

This is a generalization of the inequality obtained by S. Gatti [Metron **18** (1956), no. 1–2, 181–188; MR **18**, 683] in the special case when $r=2$. *Benjamin Epstein* (Stanford, Calif.)

6743:

De Novellis, M. Some applications and developments of Gatti-Birnbaum inequality. Metron **19** (1958), no. 1–2, 245–247.

Let $a_1, a_2, \cdots, a_n$ be real numbers and let $a=\sum_{i=1}^n a_i/n$. For any $r\geqq 1$, define

$$A_r=n^{-1}\sum_{i=1}^n |a_i-\bar{a}|^r,\ B_r=(\tfrac{1}{2}n)^{r-1}(\sum_{i=1}^n |a_i-\bar{a}|)^r.$$

It is proved that $B_{r+1}/A_{r+1}>B_r/A_r$. The monotonic increase of B_r/A_r with r is verified numerically using demographic data. *Benjamin Epstein* (Stanford, Calif.)

6744:

Shenton, L. R. Moment estimators and maximum likelihood. Biometrika **45** (1958), 411–420.

The author continues his study of the method of moments to estimate the parameters of a d.f. [Biometrika **37** (1950), 111–116; MR **12**, 193]. Using an expansion based on a moment approximation to the likelihood equation, expressions for the asymptotic covariances of the estimators are developed. An application to the negative binomial distribution is given in some detail. *I. Olkin* (Stanford, Calif.)

6745:

Watson, G. S. On chi-square goodness-of-fit tests for continuous distributions. J. Roy. Statist. Soc. Ser. B **20** (1958), 44–72.

Let $f(x, \theta_1, \cdots, \theta_s)$ be a probability density function where $\theta_1, \cdots, \theta_s$ are parameters. For a random sample $(x_1, \cdots, x_N)$ from a distribution with this density function let $\hat\theta_1, \cdots, \hat\theta_s$ be maximum likelihood estimates of $\theta_1, \cdots, \theta_s$. For specified values of p_i, $i=1, 2, \cdots, k$, let $\hat z_0(=-\infty)<\hat z_1<\cdots<\hat z_k(=+\infty)$ be chosen so that

$$(1)\qquad p_i=\int_{\hat z_{i-1}}^{\hat z_i} f(x, \hat\theta_1, \cdots, \hat\theta_s)ds,$$

where $z_i=z_i(\hat\theta_1, \cdots, \hat\vartheta_s)$. When $\hat\theta_1, \cdots, \hat\theta_s$ are replaced by $\theta_1, \cdots, \theta_s$ in (1), $\hat z_i$ will be denoted by z_i. Let $n_1, \cdots, n_k$ be the number of values of $(x_1, \cdots, x_n)$ which fall in the intervals $(\hat z_0, \hat z_1), \cdots, (\hat z_{k-1}, \hat z_k)$, respectively.

The author shows that, under certain mild conditions on $f(x, \theta_1, \cdots, \theta_k)$, the limiting distribution of the statistic $\sum_{i=1}^k (n_i-Np_i)^2/Np_i$ as $N\to\infty$ is the same as that of $\lambda_1 y_1^2+\cdots+\lambda_k y_k^2$, where $y_1, \cdots, y_k$ are independent normal random variables $N(0, 1)$, while $\lambda_1, \cdots, \lambda_k$ are the latent roots of the matrix $\|\delta_{ij}p_i\|^{-1}\cdot\|m_{ij}\|$ (δ_{ij} being the

Kronecker δ), where

$$m_{ij}=\delta_{ij}-p_ip_j-\int_{z_{i-1}}^{z_i}\int_{z_{j-1}}^{z_j} Q\,dx_1dx_2,$$

$$Q=\sum_{\alpha,\beta=1}^{k} I^{\alpha\beta}\frac{\partial f(x_1,\theta_1,\cdots,\theta_s)}{\partial\theta_\alpha}\;\frac{\partial f(x_2,\theta_1,\cdots,\theta_s)}{\partial\theta_\beta},$$

$$I^{\alpha\beta}=\left\|\mathscr{E}\left(\frac{\partial\log f(x,\theta_1,\cdots,\theta_s)}{\partial\theta_\alpha}\cdot\frac{\partial\log f(x,\theta_1,\cdots,\theta_s)}{\partial\theta_\beta}\right)\right\|^{-1}.$$

The author discusses the connection between this result and various closely related chi-square goodness-of-fit tests for continuous distributions having unknown parameters which have been obtained by Fisher, Chernoff and Lehmann, Mann and Wald, Neyman, Barton, Watson, and Wilks. *S. S. Wilks* (Princeton, N.J.)

6746:

Hodges, J. L., Jr. Fitting the logistic by maximum likelihood. Biometrics 14 (1958), 453–461.

The binomial logistic model for describing quantal response to a graded stimulus states that the probability of response at stimulus level x_i is of the form $p_i(\alpha,\beta)=[1+e^{-\alpha-\beta x_i}]^{-1}$. The problem of fitting such a function is that of estimating the parameters α and β from the number of responses r_i in n_i trials at level x_i, $i=1,\cdots,k$. Worcester and Wilson, and Berkson have used the method of maximum likelihood for making such estimates. Straightforward solutions of the equations in α and β to obtain the likelihood estimates $\hat\alpha$ and $\hat\beta$ (when they exist) involve tedious iterative procedures. In the present paper the author presents a procedure which he calls the transfer method for simplifying the solution of the maximum likelihood equations. The procedure utilizes the fact that $\sum_{i=1}^k r_i$ and $\sum_{i=1}^k r_ix_i$ constitute a set of minimal sufficient statistics for α and β, from which it follows that if numbers r_i', satisfying $\sum_{i=1}^k r_i=\sum_{i=1}^k r_i'$ and $\sum_{i=1}^k x_ir_i=\sum_{i=1}^k x_ir_i'$, are chosen so that (x_i, l_i') are collinear, where $l_i'=\log r_i'/(n_i-r_i')$, then the values of α and β satisfy $l_i'=\hat\alpha+\hat\beta x_i$, $i=1,\cdots,k$. The numerical analysis involved in determining the r_i' so that (r_i', l_i') are collinear is a fairly straightforward iterative procedure. The procedure is illustrated with three examples.

S. S. Wilks (Princeton, N.J.)

6747:

Quandt, Richard E. The estimation of the parameters of a linear regression system obeying two separate regimes. J. Amer. Statist. Assoc. 53 (1958), 873–880.

Consider the pairs of observations (X_i, Y_i), $i=1,\cdots,T$, where

$$Y_i=a_1X_i+b_1+u_{i1},\quad i\le t,$$

$$Y_i=a_2X_i+b_2+u_{i2},\quad i>t$$

$(1\le t\le T)$, and where u_{i1}, u_{i2} are normally and independently distributed with means zero and with variances σ_1^2, σ_2^2, respectively. The problem studied is that of estimating the seven unknown parameters, a_i, b_i, σ_i^2 $(i=1, 2)$ and t. The method proposed is to estimate the first six by standard methods for all admissible t and then calculate the logarithm of the likelihood ($\log L$) as a function of t. $\hat t$ is taken as that value of t which maximizes $\log L(t)$ absolutely. Two approximate tests of the hypothesis that there is only one regression are suggested and a numerical example is given.

D. G. Chapman (Seattle, Wash.)

6748:

Johnson, N. L. The mean deviation, with special reference to samples from a Pearson type III population. Biometrika 45 (1958), 478–483.

Because of the occasional outlying observations that occur in samples from leptokurtic populations, the mean deviation m will have a smaller sampling variance than the standard deviation s. This is demonstrated for Type III and Type VII populations and the factor required to make m an unbiased estimator is computed. The latter factor is, for large samples, within about 6% of its "normal" value of 0.798, for the cases considered.

G. S. Watson (Princeton, N.J.)

6749:

Correa Pólit, Héctor. Statistical inference about the parameters of nonnormal populations (confidence intervals). Trabajos Estadist. 9 (1958), 118–140. (Spanish)

Under quite general circumstances, averages of (random) groups of homogeneous observations will be more nearly normal than the observations themselves. This suggests variants of standard statistical techniques, especially those relating to population averages, when non-normality is feared: namely, replace a set of homogeneous observations by a smaller number of averages formed from (random) grouping followed by averaging, and then proceed as if these averages were the observations. The present paper applies this idea to getting confidence intervals for the population mean, with both single sample and Stein two sample designs. It also proposes similar intervals for finite populations, but only by considering a finite population as itself a large random sample, or (in the stratified case) as several random samples, one for each stratum, from populations whose variances have known ratios. There seems to be little discussion in the paper about the appropriate group size to use before taking averages.

W. Kruskal (Chicago, Ill.)

6750:

Radner, R. Minimax estimation for linear regressions. Ann. Math. Statist. 29 (1958), 1244–1250.

Two important features are not conveyed by the title. Not necessarily normal distributions are envisaged, but it is shown that, for the models considered, normal distributions are the worst the statistician has to face, in that they enforce the minimax. Again, going beyond what is usual, it is admitted that the shape, not merely the scale, of the covariance structure is largely unknown. There is no solution of bounded risk unless certain constraints on the possible values of the covariance structure is assumed; two general schemes of such assumptions are considered.

Study of the conclusions sheds light on some of the peculiarities and inadequacies of the minimax rule.

L. J. Savage (Rome)

6751:

Ferguson, Thomas. A method of generating best asymptotically normal estimates with application to the estimation of bacterial densities. Ann. Math. Statist. 29 (1958), 1046–1062.

Various minimum chi-squared methods and methods for generating B.A.N. estimates are summarized, and a new method which generates B.A.N. estimates as roots of certain linear forms is introduced and investigated. As an application of the method, the estimation of the bacterial density in an experiment using a dilution series is considered. In this case the estimator may be slightly better than those given by maximum likelihood and minimum chi-square. *G. S. Watson* (Princeton, N.J.)

6752:

Srivastava, A. B. L. Effect of non-normality on the power function of t-test. Biometrika **45** (1958), 421–430.

The power of the one-sided t-test is examined for near-normal populations described by the first four terms of an Edgeworth series, by a technique that follows from the work of Gayen. From the practical point of view, the computed power functions (for a test using the tabular significance level and a sample of 10) show that, for $|\lambda_4|\leqq 2$, $|\lambda_3|\leqq .6$, λ_4 has no effect and that the influence of λ_3 is small except in the region of the null hypothesis.

G. S. Watson (Princeton, N.J.)

6753:

Pratt, J. W. Admissible one-sided tests for the mean of a rectangular distribution. Ann. Math. Statist. **29** (1958), 1268–1271.

"Theorem. Suppose we have a sample of $n>1$ independent observations from a uniform distribution with unknown mean θ and known range R. Suppose we wish to test $H_0:\theta\leqq\theta_0$ against $H_1:\theta>\theta_0$. Then an essentially complete class of admissible tests is the class $\mathfrak{A}$ of all tests of the following type. Let u be the minimum observation, v the maximum. Let $g(u)$ be a non-increasing function of u such that $g(u)=\theta_0+\frac{1}{2}R$ for $u<\theta_0-\frac{1}{2}R$. Accept H_0 if and only if $v<g(u)$." (From the opening of the paper)

L. J. Savage (Rome)

6754:

Whittle, P. A multivariate generalization of Tchebichev's inequality. Quart. J. Math. Oxford Ser. (2) **9** (1958), 232–240.

Multivariate extensions of Chebyshev's inequality, substantially equivalent to those obtained by I. Olkin and J. W. Pratt [Ann. Math. Statist. **29** (1958), 226–234; MR **20**#385]. *Z. W. Birnbaum* (Seattle, Wash.)

6755:

Chernoff, Herman; and Savage, I. Richard. Asymptotic normality and efficiency of certain nonparametric test statistics. Ann. Math. Statist. **29** (1958), 972–994.

Suppose $x_{(1)}<\cdots<x_{(m)}$ and $y_{(1)}<\cdots<y_{(n)}$ are the order statistics of samples of sizes m and n from absolutely continuous distribution functions $F(x)$ and $G(x)$, respectively. Let $z_{Ni}=1$, $i=1, \cdots, N(N=m+n)$, if the ith smallest of the combined order statistics of the two samples is an x and 0 otherwise. Wald and Wolfowitz, Hoeffding, Lehmann, Madow, Noether and others have considered two-sample non-parametric test statistics of form

$$T_N=\sum_{i=1}^{N} c_{Ni}z_{Ni},$$

where the c_{Ni}, $i=1, \cdots, N$, are constants. Dwass has shown under certain regularity conditions on the c_{Ni}, $F(x)$ and $G(x)$, that T_N is asymptotically normally distributed for large m and n. This paper extends some of the result of these authors to more general situations. In particular, they show that Terry's c_1-statistic is asymptotically normal for large samples — a result not covered by Dwass' theorem. Hodges and Lehmann conjectured that the c_1-statistic is as efficient as the Student t-test against normal alternatives and at least as efficient as the t-test against other alternatives. This conjecture is verified in the present paper.

S. S. Wilks (Princeton, N.J.)

6756:

Dronkers, J. J. Approximate formulae for the statistical distributions of extreme values. Biometrika **45** (1958), 447–470.

This paper supplements results obtained by Fisher and Tippett, Gumbel, von Mises, de Finetti, and others on the distribution theory of extreme values in samples from continuous distribution functions. More precisely, suppose $x_{(n-m+1)}$ is the mth largest component in a sample of size n from a continuous (cumulative) distribution function $F(x)$, and let its density function be $M_{m,n}(x)$. Under certain regularity conditions on $F(x)$, including the existence of its first three derivatives, the author examines the behavior of the value of x (called $u_{m,n}$) for which $M_{m,n}(x)$ is a maximum as a function of m and n, and also the behavior of $M_{m,n}(u_{m,n})$ as a function of m and n. He gives asymptotic formulas for $u_{m,n}$ and $M_{m,n}(u_{m,n})$ for large n and fixed m. He also gives approximate formulas for $M_{m,n}(x)$ under several sets of conditions on m and n. *S. S. Wilks* (Princeton, N.J.)

6757:

Bejar, Juan. Contingency tables. Trabajos Estadist. **9** (1958), 85–101. (Spanish. English summary)

This paper presents the standard chi-square test for independence in an r-way contingency table. It also presents the standard chi-square test for independence between disjoint groups of classifications; this test reduces immediately to that relating to single classifications by coalescing the classifications of each group into a single product classification. The chi-square tests discussed here are well known, but seldom explicitly described. However, the chi-square test statistic for independence in an r-way contingency table was explicitly stated by Karl Pearson [Biometrika **11** (1915/17), 145–158], but with an erroneous instruction about degrees of freedom, and without an argument for the customary so-called "expected" frequencies. These two difficulties were cleared up in general terms by later publications of R. A. Fisher and others [see, for example, papers 5 and 7 of Fisher's "Contributions to Mathematical Statistics", Wiley, New York, 1950; MR **12**, 427], but the reviewer knows of no full explicit statement in print for the case of r-way contingency table independence prior to Bejar's paper. *W. Kruskal* (Chicago, Ill.)

6758:

van Elteren, Ph. The asymptotic distribution for large m of Terpstra's statistic for the problem of m rankings. Nederl. Akad. Wetensch. Proc. Ser. A **60**=Indag. Math. **19** (1957), 522–534.

The Friedman [J. Amer. Statist. Assoc. **32** (1937), 675–699] analysis of variance statistic is essentially the average of Spearman rank correlations. The Terpstra statistic [Nederl. Akad. Wetensch. Proc. Ser. A **58** (1955), 690–696; **59** (1956), 59–66; MR **17**, 983] is the corresponding average of Kendall's tau. In this paper, the large sample distribution of the Terpstra statistic is shown to be a mixture of two independent central chi variables when (a) the null hypothesis is true, and (b) tied observations can frequently occur.

I. R. Savage (Minneapolis, Minn.)

6759:

Freeman, G. H. Families of designs for two successive experiments. Ann. Math. Statist. **29** (1958), 1063–1078.

The author tabulates all known families of designs with

two associate classes in the setting where two successive experiments are conducted on the same plots, the first set of treatments being orthogonal with respect to blocks, and the second set being partially balanced both with respect to blocks and also with respect to the first set.

J. Kiefer (Oxford)

6760:

Mihalevič, V. S. Sequential selection between two solutions for a Poisson process. Teor. Veroyatnost. i Primenen. 3 (1958), 465–470. (Russian. English summary)

Continuation of the author's work [Teor. Veroyatnost. i Primenen. 1 (1956), 437–465; MR 19, 694]. The author appears to have overlooked the paper by Dvoretzky, Kiefer, and Wolfowitz [Ann. Math. Statist. 24 (1953), 254–264; MR 14, 997, 1279].

J. Wolfowitz (Ithaca, N.Y.)

6761:

Hall, Wm. Jackson. Most economical multiple-decision rules. Ann. Math. Statist. 29 (1958), 1079–1094.

$\{X_i\}$ is a sequence of independent, identically distributed random variables each having a density function belonging to a specified class Ω. The decision problem considered in this paper is to formulate a decision rule (d.r.) for choosing a non-random positive integer n and, after taking an observation on $X_1, \cdots, X_n$, for choosing one of m possible alternative decisions $A_1, \cdots, A_m$. The author first considers simple discrimination where Ω contains only a finite number, l, of elements, f_i. A d.r. D is characterized by the functions $p_{ij}(D)=\text{prob}$ (D chooses $A_j|f_i$). If $l=m$, D_j is correct for f_j and $\alpha=(\alpha_1, \cdots, \alpha_m)$ is a vector of positive constants each less than one, a d.r. D_N, based on a sample of size N, is said to be a most economical d.r. relative to α if it satisfies $p_{ii}(D)\geqq\alpha_i$ $(i=1, \cdots, m)$ and N is the least integer for which this may be satisfied for some D_n based on a sample of size n. If $\beta=(\beta_{ij})$ is a matrix of positive constants with $\beta_{ij}=1$ if A_j is correct for f_i, a d.r. D_N is said to be a most economical d.r. relative to β if it satisfies $p_{ij}(D)\leqq\beta_{ij}$ $(i=1, \cdots, l;\ j=1, \cdots, m)$ and N is the least integer for which this may be satisfied for some D_n based on a sample of size n. Loosely, the probabilities of error are to be bounded and one requires the smallest sample that will do this. In the vector case the author introduces a loss function $W_{ij}=-\delta_{ij}/\alpha_i$ and shews that the most economical d.r. is minimax with respect to this loss function, and the sample size is the least which is such that the minimax value is not greater than -1. Such a d.r. is a likelihood ratio rule and, in general, the components of the least favourable distribution are positive. An artificial decision problem is introduced in order to develop similar results in the matrix case, again giving minimax procedures but unlikelihood ratio d.r.'s. These results are extended in a final section to composite discrimination where Ω is not finite. Disjoint subsets $w_1, \cdots, w_l$ of Ω are given: $\inf_{\theta\in w_i} p_i(\theta, D)$ replaces $p_{ii}(D)$, for example.

D. V. Lindley (Cambridge, England)

6762:

Mendenhall, William; and Hader, R. J. Estimation of parameters of mixed exponentially distributed failure time distributions from censored life test data. Biometrika 45 (1958), 504–520.

This paper attempts to generalize the estimation problem to the case where the underlying c.d.f. of life is given by the mixture

$$F(t)=pF_1(t)+(1-p)F_2(t),$$

where $0\leqq p\leqq 1$ and $F_i(t)=1-e^{-t/\alpha_i}$, $i=1, 2$ (in the strictly exponential case, $p=1$ and $\alpha_1=\alpha_2=\alpha$). The parameters p, α_1, and α_2 are unknown. The method of maximum likelihood is used to obtain point estimates of p, α_1, and α_2 from life tests, where n items are placed on test initially and where the life test is terminated after a fixed time T has elapsed. Small sample properties of the estimates were investigated by means of empirical sampling. As might be expected, maximum likelihood procedures give good estimates when the sample size n is large and when the stopping time T is large relative to the mean lives α_1 and α_2. When n and T are small, the estimates obtained by maximum likelihood are badly biased and have large standard deviations. *Benjamin Epstein* (Stanford, Calif.)

6763:

Mendenhall, William. A bibliography on life testing and related topics. Biometrika 45 (1958), 521–543.

This bibliography contains more than 600 references to articles having either a direct, indirect, or in some cases peripheral bearing on the design and analysis of life tests. Those having specialized interests will have to screen this bibliography, using a fine or wide mesh screen depending on the degree of specialization (or generality) involved. The references are classified into the following nine categories: (1) Censored or truncated sampling: (2) renewal theory and its application; (3) order statistics; (4) theory of extreme values; (5) papers concerned with failure rates, conditional failure density, tests for randomness of events occurring in time; (6) fatigue testing; (7) machine productivity problems; (7) system reliability; (8) analysis of sensitivity data and fitting of dosage mortality curves. *Benjamin Epstein* (Stanford, Calif.)

6764:

Ishii, Goro. Kolmogorov-Smirnov test in life test. Ann. Inst. Statist. Math. 10 (1958), 37–46.

The author develops a truncated Kolmogorov-Smirnov test, which can be used as a non-parametric two-sample life test. Testing is terminated when a preassigned total number of failures has occurred and the rule of action is based on the maximum distance between the truncated sample c.d.f.'s. As is usual in problems of this kind, the formulae obtained involve counting paths. The application of these results to actual data presents formidable computational difficulties. For a related two-sample non-parametric life test which is also based on a truncation of the Kolmogorov-Smirnov procedure one should see C. K. Tsao [Ann. Math. Statist. 25 (1954), 587–592; MR 16, 270]. *Benjamin Epstein* (Stanford, Calif.)

6765:

Flehinger, B. J.; and Lewis, P. A. Two-parameter lifetime distributions for reliability studies of renewal processes. IBM J. Res. Develop. 3 (1959), 58–73.

If T=life-length of a component is a random variable with the distribution function $F(t)=\text{Prob}\,\{T\leqq t\}$, the "hazard" is customarily defined as $z(t)=F'(t)/[1-F(t)]$. The authors study the following two-parametric families of life-distributions: (1) those with $z(t)=a+2b^2t$, $0<a<\infty$, $0<b<\infty$; (2) those with $z(t)=a+3c^3t^2$, $0<a<\infty$, $0<c<\infty$; and (3) those with the normal distribution $N(t_0;\sigma^2)$ truncated to $t\geqq 0$. Families (1) and (2) contain as limiting cases the exponential and the Weibull distributions. For each of these families, probability density, survival probability, expected number of replacements and renewal rate are computed numerically and presented

graphically for a number of examples. An entropic measure of randomness is proposed, to reflect such facts as that, intuitively, the life-lengths seem to be "completely random" for the exponential distribution and not random at all for a degenerate distribution ascribing probability 1 to one value of T.

Z. W. Birnbaum (Seattle, Wash.)

6766:

Scheffé, Henry. Experiments with mixtures. J. Roy. Statist. Soc. Ser. B. **20** (1958), 344–360.

A theory is developed for experiments with mixtures of q components whose purpose is the empirical prediction of the response to any mixture of the components, when the response depends only on the proportion of the components and not on their total amount. It is assumed that the response is a polynomial function of the proportions of the components. Designs are suggested for fitting, and testing the goodness-of-fit of, these polynomials. Many interesting questions are raised.

G. S. Watson (Princeton, N.J.)

6767:

Weinstein, Abbott S. Alternative definitions of the serial correlation coefficient in short autoregressive sequences. J. Amer. Statist. Assoc. **53** (1958), 881–892.

Various estimators for the serial correlations of small lags in a stationary series with an unknown mean and variance are tested on an artificially constructed second-order autogressive process, using samples of 14. It is shown that the method of eliminating the unknown mean is more crucial than the method for eliminating the unknown variance, as has been found in comparable problems.

G. S. Watson (Princeton, N.J.)

NUMERICAL METHODS

See also 6805, 6806, 6807, 6947.

6768:

★Hartree, D. R. Numerical analysis. 2nd ed. Oxford University Press, New York, 1958. xvi+302 pp. $6.75.

The main change from the first edition [Clarendon Press, Oxford, 1952; MR **14**, 690] is in the chapter on the numerical integration of differential equations, which has been rearranged and extended, particularly in the treatment of equations with two-point boundary conditions. Amongst other additions are: a section on Whittaker's 'cardinal function' in the theory of interpolation; an account of Wilkes' method of handling the Choleski method of matrix factorization; an extension of the treatment of Gaussian quadrature; the brief introduction to digital computers is omitted, since works on this subject are now readily available.

The book retains its character as an excellent introduction to numerical analysis for those interested in applying them to practical problems.

6769:

Brown, O. E. Computation of common logarithms by repeated squarings. Amer. Math. Monthly **65** (1958), 118–120.

The author observes that the following algorithm is readily programmed:

If $1 \leqq N_1 < 10$, and

$$a_i = \begin{cases} 0 \\ 1 \end{cases} \text{ while } N_{i+1} = \begin{cases} N_i^2 & \text{if } N_i^2 < 10 \\ N_i^2/10 & \text{if } N_i^2 \geqq 10, \end{cases}$$

then $\log_{10} N_1 = \sum_{i=1}^{\infty} 2^{-i} a_i$.

S. Gorn (Philadelphia, Pa.)

6770:

Nečepurenko, M. A letter to the editor. Dokl. Akad. Nauk SSSR **123** (1958), 214. (Russian)

It is noted that an additional hypothesis is needed for some of the theorems in an earlier paper of the author [same Dokl. **109** (1956), 704–706; MR **18**, 235].

R. G. Bartle (Urbana, Ill.)

6771:

Bass, J.; et Guilloud, J. Méthode de Monte-Carlo et suites uniformément denses. Chiffres **1** (1958), 149–155.

For Monte Carlo estimates of definite integrals, the authors consider the use of sequences uniformly dense on the interval of integration [H. Weyl, Math. Ann. **77** (1916), 313–352], and present several such. For the integral

$$I = \int_0^1 (1-t^2)^{\frac{1}{2}} \cos(xt)\,dt$$

they compare the result obtained with a uniformly dense sequence and that obtained using a sequence of "nombres aléatoires", to the apparent advantage of the former method.

A. S. Householder (Oak Ridge, Tenn.)

6772:

★Buck, R. Creighton. Survey of recent Russian literature on approximation. On numerical approximation. Proceedings of a Symposium, Madison, April 21–23, 1958, pp. 341–359. Edited by R. E. Langer. Publication no. 1 of the Mathematics Research Center, U. S. Army, the University of Wisconsin. The University of Wisconsin Press, Madison, 1959. x+462 pp. (1 insert) $4.50.

This important survey article gives a good picture of what is going on today in the field of approximation theory, for the author has wisely tied in the Russian papers with work that has appeared outside that country. Five principal areas are covered: approximation on a bounded interval, approximation on the whole axis, approximation by interpolation polynomials, approximation of functions of several variables, and approximation of analytic functions. The bibliography contains over 135 items and most of them are given with the appropriate Mathematical Reviews reference.

P. Davis (Washington, D.C.)

6773:

Dent, Benjamin A.; and Newhouse, Albert. Polynomials orthogonal over discrete domains. SIAM Rev. **1** (1959), 55–59.

Dans la détermination, par la méthode des moindres carrés, d'une courbe passant aussi près que possible d'un système de n points, on utilise un système de n polynomes orthogonaux. Dans un travail récent [Philos. Mag. **41** (1950), 124–137; MR **11**, 692], Guest donne une méthode indirecte pour le calcul des coefficients de ces polynomes. La méthode donnée ici est immédiate et utilise les matrices. Une application numérique est faite avec trois points.

R. Campbell (Caen)

6774:

Rivlin, T. J. Smooth interpolation. SIAM Rev. **1** (1959), 60–63.

Given n distinct points (x_i, y_i), $i=1, \cdots, n$, there is a

unique polynomial of degree n, $P(x)$, which passes through these points and for which the arc length

$$\int_{x_1}^{x_n}(1+(P'(x))^2)^{\frac{1}{2}}dx$$

is minimum. The author calls this polynomial a "smooth" interpolant, and determines it in terms of the zero of a certain transcendental equation. Several examples are presented which were computed from an IBM 650 program. *P. Davis* (Washington, D.C.)

6775:

Mosteller, Frederick; and Richmond, D. E. Factorial $\frac{1}{2}$: a simple graphical treatment. Amer. Math. Monthly **65** (1958), 735–742.

An expository treatment of the definition of factorial $\frac{1}{2}$ due to Gauss. *W. H. Muller* (The Hague)

6776:

Novák, Mirko. A discussion of some methods of calculation of the function sn (u, k). Apl. Mat. 3 (1958), 401–427. (Czech. Russian and English summaries)

After explaining that elliptic functions are of importance in electrical network theory, in particular in connection with frequency filters, the author gives a brief account of the mathematical properties of $\mathrm{sn}(u, k)$. He describes the numerical tables available for this function, explains their use, and notes that none appear to be completely satisfactory for computations of electrical networks. He then discusses in greater detail the computation of $\mathrm{sn}(u, k)$ by means of theta functions and from the Fourier expansion. Graphs are given for a number of auxiliary functions; these graphs may be used to determine the number of terms which must be used in order to compute $\mathrm{sn}(u, k)$ to a preassigned accuracy.
A. Erdélyi (Pasadena, Calif.)

6777:

★Faddeeva, V. N. Computational methods of linear algebra. Translated by C. D. Benster. Dover Publications, Inc., New York, 1959. xi+252 pp. $1.95.

This book is a very readable translation of the Russian work, *Vyčislitel'nye metody lineĭnoĭ algebry*, Gosudarstv. Izdat. Tehn.-Teoret. Lit., Moscow-Leningrad, 1950 [MR 13, 872].

6778:

Bauer, F. L. Sequential reduction to tridiagonal form. J. Soc. Indust. Appl. Math. 7 (1959), 107–113.

The author is interested in the explicit representation of certain matrices $T_i=\begin{pmatrix} I_i & 0 \\ 0 & Q_i \end{pmatrix}$ and their inverses T_i^{-1}, where I_i is the ith order identity, such that, if $A=A_0$ and $A_{i+1}=T_i^{-1}A_iT_i$, then A_{n-1} is tridiagonal in form. Several such matrices are exhibited and the numerical stability of the computation is discussed.
A. S. Householder (Oak Ridge, Tenn.)

6779:

Stiefel, E. Recent developments in relaxation techniques. Proceedings of the International Congress of Mathematicians, Amsterdam, 1954, Vol. 1, pp. 384–391. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam; 1957. 582 pp. $7.00.

The author discusses several linear iterative methods for solving the system of linear equations $Ax=k$. He considers in detail "scalar" iteration schemes defined by

$$\Delta x_{i+1}=A_ir_i+C_i\Delta x_i+D_i\Delta x_{i-1}+\cdots,$$

where A_i, C_i, D_i, are scalars, where $\Delta x_i=x_i-x_{i-1}$, and where x_i denotes the ith approximation to the exact solution $x=A^{-1}k$. For convenience x_0 is assumed to be zero. For cases where A is symmetric and positive definite, a "strategy" is developed for choosing among algorithms of the class with a given number of steps, n, the one yielding x_n closest to the true solution. Residual polynomials $R_i(A)$ in the matrix A are considered such that $R_i(0)=1$ and such that $r_i=R_i(A)k$, where the residual r_i is given by $k-Ax_i$. Assuming, without loss of generality, that unity is an upper bound for all of the eigenvalues of A, the author seeks to minimize the quantity

$$\psi_i=\int_0^1 [R_i(\lambda)/\lambda]\rho(\lambda)d\lambda,$$

where $\rho(\lambda)$ is an arbitrary density function in the interval $0\leqq\lambda\leqq 1$. It can be shown that in the family of nth degree polynomials which satisfy the condition $R_n(0)=1$, the error measure, ψ, takes on a minimum value for the $(n+1)$-st polynomial of the orthogonal set belonging to the density function $\rho(\lambda)$. Because of the fact that every three successive polynomials of the orthogonal set are related by a recursion formula, the process of best strategy involves carrying out an iterative process of second order using coefficients from the recursion formula. There is no need to consider procedures of order higher than second.

For the case where no information about the eigenvalues is known except that they lie in the range $0<\lambda\leqq 1$ the author discusses the density function $\rho(\lambda)=\lambda^\alpha(1-\lambda)^\beta\cdot f(\lambda)$, $\alpha>0$, $\beta>-1$, where $f(\lambda)$ has a continuous second derivative over the range $0\leqq\lambda\leqq 1$ and satisfies $0<c_1\leqq f(\lambda)\leqq c_2$. In the case $f(\lambda)\equiv 1$ the hypergeometric polynomials are obtained for the residual polynomials.

If a positive lower bound, ε, for the eigenvalues is known, then one is led to consider Tschebyscheff polynomials in the interval $\varepsilon\leqq\lambda\leqq 1$. The resulting procedure is related to a first order procedure which is basically Richardson's method [Philos. Trans. Roy. Soc. London, Ser. A **210** (1911), 307–357], as adapted for use with Tschebyscheff polynomials by G. Shortley [J. Appl. Phys. **24** (1953), 392–396; MR **14**, 1019] and by D. Young [J. Math. Phys. **32** (1954), 243–255; MR **15**, 650]. The first order and the second order procedures both yield the same results after n iterations, even though intermediate results are different.

If one chooses

$$\rho(\lambda)=\sum_i k_i^2\delta(\lambda-\lambda_i),$$

where δ is the Dirac function and where k_j is the jth component of k, then $\psi_i=(Ay_i, y_i)$, where $y_i=A^{-1}k-x_i$. In this case one is led to the conjugate gradient method of Hestenes and Stiefel [J. Res. **49** (1952), 409–436; MR **15**, 651]. *D. M. Young, Jr.* (Austin, Tex.)

6780:

Semarne, H. Manvel. New direct method of solution of a system of simultaneous linear equations. SIAM Rev. **1** (1959), 53–54.

The system $\sum_{j=1}^n a_{ij}x_j=c_i$, $i=1, 2, \cdots, n$, is solved by finding the column vector $(x_1, x_2, \cdots, x_n, 1)$ orthogonal to the n row vectors $e_i=(a_{i1}, a_{i2}, \cdots, a_{in}, -c_i)$. Using the Gram-Schmidt process for constructing an orthogonal set, and writing (y, z) for the scalar product of the vectors y and z and $e_{n+1}=(1, 1, \cdots, 1, 1)$, there is

formed:

$$f_1=e_1;\ f_2=e_2-\frac{(f_1,e_2)}{(f_1,f_1)}f_1;\ \cdots;$$

$$f_{n+1}=e_{n+1}-\frac{(f_1,e_{n+1})}{(f_1,f_1)}f_1-\cdots-\frac{(f_n,e_{n+1})}{(f_n,f_n)}f_n.$$

The solution is the vector f_{n+1}.

C. C. Gotlieb (Toronto, Ont.)

6781:

Newman, Morris; and Todd, John. The evaluation of matrix inversion programs. J. Soc. Indust. Appl. Math. 6 (1958), 466–476.

The authors point out how difficult and tedious it is to undertake error estimates for significant sized calculations. They, therefore, suggest that a practical technique is to undertake "experimental" calculations on "representative" problems in which the exact results are known. With the help of these experiments, it is possible, more or less, to extrapolate to real situations. They illustrate their ideas with matrix inversion. They deal with a class of matrices whose exact inverses have been obtained and for which the conditions of the matrices can be well estimated. They indicate the results of experiments performed on various machines. The results are compared to the theoretical error computations made by von Neumann and Goldstine.

H. H. Goldstine (Princeton, N.J.)

6782:

Lomont, J. S.; and Willoughby, R. A. Dominant eigenvectors of a class of test matrices. SIAM Rev. 1 (1959), 64–65.

This is a solution of a matrix problem of elementary particle physics. An irreducible set $\boldsymbol{\sigma}=(\sigma_x,\sigma_y,\sigma_z)$ of three Hermitian matrices satisfying the commutation rules $[\sigma_x,\sigma_y]=i\sigma_z$ (cyclic) is given. Formulas are derived for the eigenvalues and the normalized dominant eigenvectors of the matrix $\mathbf{k}\cdot\boldsymbol{\sigma}=k_x\sigma_x+k_y\sigma_y+k_z\sigma_z$, where $\mathbf{k}$ is a real vector.

P. S. Dwyer (Ann Arbor, Mich.)

6783:

White, Paul A. The computation of eigenvalues and eigenvectors of a matrix. J. Soc. Indust. Appl. Math. 6 (1958), 393–437.

This paper is a very careful exposition of the known methods for the computation of eigenvalues and eigenvectors of matrices, both real and complex. The author first discusses Jacobi's method and several variants thereon; the Givens method and the gradient method. All of these are discussed for the real symmetric case. He then discusses the case in which the matrix is real and non-symmetric and presents the power method and a variant of that; Wilkinson's method; Wielandt's method and various deflation techniques. Finally, he takes up the general complex case and outlines Greenstadt's method, Lanczos' method and modification of that method. He discusses various methods for evaluating the characteristic polynomial. Among those he reviews are Givens' method, Hyman's method, Krylov's method, Danilewsky's method, Le Verrier's method, Souriau-Frame's method and lastly, Leppert's method.

H. H. Goldstine (Princeton, N.J.)

6784:

Fröberg, Carl-Erik. Diagonalization of Hermitian matrices. Math. Tables Aids Comput. 12 (1958), 219–220.

The author extends Jacobi's method for the diagonalization of real symmetric matrices to Hermitian ones. In this case, of course, the transformation matrices are unitary. He indicates that the technique has been applied on SMIL, the electronic machine at Lund University, for matrices up to order 15. The length of the calculation came out to about four times longer than that for the real symmetric case.

H. H. Goldstine (Princeton, N.J.)

6785:

Horák, Vladimír. Zu einer Lösungsmethode der algebraischen Gleichungen mit vielen komplexen Wurzelpaaren nach dem Graeffeschen Verfahren. Časopis Pěst. Mat. 82 (1957), 440–453. (Czech. Russian and German summaries)

Der Verfasser untersucht die folgende bekannte Methode der Lösung einer reellen algebraischen Gleichung $f(x)=0$, $2n$-ten Grades mit lauter komplexen einfachen Wurzelpaaren: Man findet (z.B. mit Hilfe des Graeffeschen Verfahrens) die absoluten Beträge $r_1, r_2, \cdots, r_n$ $(r_1\leqq r_2\leqq\cdots\leqq r_n)$ bzw. $\rho_1, \rho_2, \cdots, \rho_n$ (jetzt schon $\rho_1<\rho_2<\cdots<\rho_n$) der Wurzelpaare der Gleichungen $f(x)=0$ bzw. $f(x+u)=0$, wo $0<u<U=\frac{1}{2}\min(r_{i+1}-r_i)$ für $i=1, \cdots, n-1$, $r_{i+1}\neq r_i$. Dann sind die Wurzeln von $f(x)=0$ gleich den Wurzeln von n Gleichungen $(i=1, \cdots, n)$

(1) $$x^2+u^{-1}(\rho_i^2-r_i^2-u^2)x+r_i^2=0.$$

Es werden Regeln angegeben, die zur Bestimmung der Anzahl von richtigen Ziffern der Koeffizienten in (1) im Zehnersystem dienen, wenn die Anzahl der richtigen Ziffern von r_i und ρ_i bekannt ist. Die Anwendung dieser Regeln (bei vorgeschriebener Genauigkeit) erfordert, die absoluten Beträge der Wurzeln r_i zuerst mit kleinerer Genauigkeit zu bestimmen.

M. Fiedler (Prague)

6786:

Coppel, William Andrew. The solution of cubic equations by iteration. Z. Angew. Math. Phys. 9a (1958), 380–383. (Italian summary)

L'auteur indique comment écrire une équation du troisième degré de manière que la convergence de la méthode d'itération soit assurée.

J. Kuntzmann (Grenoble)

6787:

***Davis, Philip J. On the numerical integration of periodic analytic functions.** On numerical approximation. Proceedings of a Symposium, Madison, April 21–23, 1958, pp. 45–59. Edited by R. E. Langer. Publication no. 1 of the Mathematics Research Center, U. S. Army, the University of Wisconsin. The University of Wisconsin Press, Madison, 1959. x+462 pp. (1 insert) \$4.50.

This paper examines the well-known effect that the trapezoid rule of integration frequently gives very good results. In particular, the paper compares the trapezoid rule with Gaussian rules. The approach is that of selecting analytic functions and studying their behavior in the complex plane; in particular, the asymptotic behavior of the error as the number of abscissas increases is examined carefully. The conclusions may be briefly summarized by saying that which type of rule has the smaller error depends on the analytic behavior of the function; singularities near the interval of integration favor the Gaussian, while a wide strip of analyticity favors the trapezoid rule.

R. W. Hamming (Murray Hill, N.J.)

6788:

***Hammer, Preston C. Numerical evaluation of multiple integrals.** On numerical approximation. Proceedings of a Symposium, Madison, April 21–23, 1958, pp. 99–115. Edited by R. E. Langer. Publication no. 1 of the Mathematics Research Center, U. S. Army, the University of Wisconsin. The University of Wisconsin Press, Madison, 1959. x+462 pp. (1 insert) \$4.50.

This is a valuable expository paper dealing with

numerical integration formulae of the form

$$\int_R w(x)f(x)dx = \sum_{j=1}^{m} a_j f(x_j),$$

where R is a region of the Euclidean space E_n ($n \geqq 2$). Special attention is given to "fully symmetrical" regions R, e.g., spheres and cubes, and to other regions which frequently occur, such as cones and simplexes. Much of the material presented in this paper has been developed in the last few years by the author himself and by his collaborators. There is a bibliography containing 36 papers.
Walter Gautschi (Washington, D.C.)

6789:

Longman, I. M. On the numerical evaluation of Cauchy principal values of integrals. Math. Tables Aids Comput. 12 (1958), 205–207.

The Cauchy principal value of an integral

$$I = P\int_{-a}^{a} f(x)dx$$

(f is assumed to have a sole singularity at $x=0$) is handled by taking the even part of f, $h(x) = \frac{1}{2}[f(x)+f(-x)]$, and dealing with the equivalent $I = 2\int_0^a h(x)dx$. The function $h(x)$ may or may not have a singularity at $x=0$. Some tips are given for dealing with this form. Three examples illustrate the method. *P. Davis* (Washington, D.C.)

6790:

Tlegenov, K. B. On mechanical solution of certain systems of linear differential equations with constant coefficients. Izv. Akad. Nauk Kazah. SSR. Ser. Mat. Meh. no. **6(10)** (1957), 87–96. (Russian. Kazah summary)

Let the n-vector Y satisfy

(1) $$Y' = AY, \quad Y(x_0) = Y_0 \quad ('=d/dx),$$

where A is a constant matrix. The solution of (1) at the points x_0+ih ($i=0, 1, 2, \cdots$) is approximated by the solution of the related difference equation

(2) $$Y_{i+1} = (E+hA)Y_i = UY_i.$$

In (2) the substitution $Z=QY$ is made, where Q is a matrix whose first row is $(1, 0, \cdots, 0)$, so that the first components of Y and Z agree. With this substitution, (2) becomes (3) $Z_{i+1} = \Omega Z_i$. The author finds that if Ω is to have a particular simple form, easily adapted to machine computation, it is necessary that the elements of A satisfy

$$a_{1j}^{-1}\sum_i a_{1i}a_{ij} = a_{1k}^{-1}\sum_i a_{1i}a_{ik} \text{ for all } (j, k),$$

$j, k = 1, 2, \cdots, n$. The size of h is then determined by the choice of Ω. The process is illustrated by an example with $n=3$. There are numerous annoying misprints in the paper, and the matrix manipulations are not always clear.
W. S. Loud (Minneapolis, Minn.)

6791:

Fehlberg, Erwin. Eine Methode zur Fehlerverkleinerung beim Runge-Kutta-Verfahren. Z. Angew. Math. Mech. 38 (1958), 421–426. (English, French and Russian summaries)

The author observes that if the initial value problem (1) $y'=f(x, y)$, $y(x_0)=y_0$ has the property (2) $y_0'=y_0''=(f_y)_0=0$, then it is possible to obtain special Runge-Kutta type formulas for its numerical solution which are of higher accuracy than the ordinary fourth order formula. The author, in fact, constructs a family of sixth order formulas each of which involves only three substitutions into the differential equation. One of these is $y_1 = y_0+(1/1152)(500k_1+375k_2+64k_3)$, where $k_1 = hf(x_0+(2/5)h,\ y_0)$, $k_2 = hf(x_0+(4/5)h,\ y_0+(4/3)k_1)$, $k_3 = hf(x_0+h,\ y_0+(5/128)(27k_2-76k_1))$. If condition (2) is not satisfied for a given differential equation (3) $\bar{y}' = \bar{f}(x, \bar{y})$, $\bar{y}(x_0) = \bar{y}_0$, then the substitution $\bar{y} = y+\bar{y}_0'(x-x_0)+\frac{1}{2}\bar{y}_0''(x-x_0)^2+(\bar{f}_{\bar{y}})_0(x-x_0)(y-\bar{y}_0)$ is proposed, which transforms (3) into an equation (1) satisfying (2). In particular

$$f(x, y) = [1+(\bar{f}_{\bar{y}})_0(x-x_0)]^{-1}[\bar{f}(x, \bar{y}(x, y))-\bar{y}_0' \\ -\bar{y}_0''(x-x_0)-(\bar{f}_{\bar{y}})_0(y-\bar{y}_0)], \quad y_0 = \bar{y}_0.$$

An analogous transformation is proposed for arbitrary nth order differential equations and Runge-Kutta formulas are constructed with correspondingly higher accuracy.
Walter Gautschi (Washington, D.C.)

6792:

Collatz, L. Einige funktionalanalytische Methoden bei der numerischen Behandlung von Differentialgleichungen. Z. Angew. Math. Mech. 38 (1958), 264–267.

Viele Probleme der praktischen Analysis lassen sich auf die Form $u=Tu$ bringen und dann mit dem Iterationsverfahren $u_{n+1}=Tu_n$ behandeln. Für solche Iterationsverfahren in metrischen und pseudometrischen Räumen sind Fehlerabschätzungen angegeben worden. Ausserdem existieren einfache Einschliessungsaussagen im Falle monotoner Operatoren. Der Verfasser berichtet über die numerische Anwendung dieser allgemeinen Sätze auf Differentialgleichungsaufgaben und teilt verschiedene numerische Beispiele mit. *J. Schröder* (Hamburg)

6793:

Collatz, Lothar. Fehlerabschätzungen bei Randwertaufgaben partieller Differentialgleichungen mit unendlichem Grundgebiet. Z. Angew. Math. Phys. **9a** (1958), 118–128.

Es wird gezeigt, wie man für Randwertaufgaben bei partiellen Differentialgleichungen mit unendlichem Grundgebiet Fehlerabschätzungen erhält, indem man diese Aufgaben durch einfache Transformationen in solche mit endlichem Grundgebiet überführt und dann die dafür gültigen Randmaximumprinzipien und ähnliche Prinzipien benutzt. Der erste Abschnitt behandelt elliptische Differentialgleichungen

$$-P(X_j)\Delta U + \sum_{k=1}^{n} Q_k(X_j)U_k + S(X_j)U = 0 \quad (U_k = \partial U/\partial X_k)$$

bei unendlichem Grundgebiet des $(X_1, X_2, \cdots, X_n)$-Raumes. Mit Hilfe der Transformation

$$x_j = X_j/R^2,\ u(x_j) = r^{2-n}U(x_j r^{-2}),\ R = \left(\sum_{k=1}^{n} X_j^2\right)^{\frac{1}{2}},\ r = 1/R$$

kommt man hierbei (unter geeigneten Voraussetzungen) für die erste und dritte Randwertaufgabe und im Falle $n>2$ auch für die zweite Randwertaufgabe zu Fehlerabschätzungen, da diese Aufgaben in die erste und dritte Randwertaufgabe bei endlichem Grundgebiet übergehen. Damit lassen sich z.B. auch räumliche Umströmungsaufgaben der Potentialtheorie behandeln. Diese Abschätzungsmethoden bilden daher in gewissem Sinne ein Gegenstück zu den Methoden der konformen Abbildung, welche bei Umströmungsaufgaben in 2, nicht aber 3 Raumdimensionen anwendbar sind. Der zweite Abschnitt behandelt parabolische Differentialgleichungen $U_t - F(X_j, t, U, U_j, U_{jj}) = 0$. Dabei werden auch allgemeinere Transformationen besprochen, insbesondere wird der Spezialfall einer Raumdimension diskutiert. Zwei Beispiele (erste und zweite Randwertaufgabe der Potentialtheorie in drei Dimensionen) erläutern die Abschätzungsmethode. *J. Schröder* (Hamburg)

6794:

Szabó, J. Ein neues Verfahren zur unmittelbaren numerischen Lösung der Dirichletschen Randwertaufgaben. Z. Angew. Math. Mech. 38 (1958), 280–284.

Der Verfasser untersucht das gewöhnliche Differenzenverfahren für bestimmte Typen von Randwertaufgaben mit partiellen Differentialgleichungen gerader Ordnung und rechteckigem Grundbereich. W_{ik} seien die Näherungswerte in den Gitterpunkten eines Rechteckgitters. Die $m\times n$-Matrix $W=(W_{ik})$ lässt sich in geschlossener Form angeben:

$$W=U_m\cdot\{M\times(U_m\cdot P\cdot U_n)\}\cdot U_n.$$

U_m ist die orthogonale $m\times m$-Matrix der Eigenvektoren einer "primitiven Kontinuantenmatrix" C_m, M enthält u.a. die Eigenwerte von C_m und C_n, die Matrix P besteht aus gegebenen Werten, und es ist $A\times B=(A_{ik}B_{ik})$ definiert. Ein numerisches Beispiel (Laplacesche Gleichung) erläutert die Anwendung der Formel.

J. Schröder (Hamburg)

6795:

Wasow, Wolfgang. On the accuracy of implicit difference approximations to the equation of heat flow. Math. Tables Aids Comput. 12 (1958), 43–55.

This paper considers the convergence, the truncation error, and the stability of difference approximations for the problem

$$L[u]=u_t-u_{xx}=0,\ 0<x<\pi,\ 0<t\leqq T,$$

where $u(x,0)=f(x)$ and $u(0,t)=u(\pi,t)=0$. The difference approximation considered is

$$L_{h,k}[U]=k^{-1}\{U(x,t+k)-U(x,t)\}-sh^{-2}\{U(x+h,t+k)$$
$$-2U(x,t+k)+U(x-h,t+k)\}-(1-s)h^{-2}\{U(x+h,t)$$
$$-2U(x,t)+U(x-h,t)\}=0,\ 0<x<\pi,\ 0<t\leqq T,$$

where π/h is an integer and $0<h<h_1<1$, $0<k\leqq k_1<1$. The results of this investigation differ from those of P. Lax and R. Richtmyer [Comm. Pure Appl. Math. 9 (1956), 267–293; MR 18, 48] in that appraisals of truncation error are pointwise or uniform rather than bounds on the mean square norm.

Convergence is shown for $f(x)=\sum_1^\infty b_r\sin rx$ integrable, providing $\sum_1^\infty|b_r|<\infty$, and $2(1-2s)k-h^2\leqq 0$, where s is not restricted to $0\leqq s\leqq 1$.

A bound for truncation error is obtained as

$$|U(x,t)-u(x,t)|\leqq ct^{-1}(h^2+k^\alpha),$$

where

$$C=M(s,p,t)\sup_{0\leqq x\leqq\pi}(|f''(x)|+|f'''(x)|)$$

and $\alpha=1$ for $s\neq\frac{1}{2}$, $\alpha=2$ for $s=\frac{1}{2}$. The preceding bound is true for $\frac{1}{2}\leqq s\leqq 1$, $h/k\geqq p>0$, $k\leqq t\leqq T$, $h\leqq h_1<p_2$, $f(0)=f(\pi)=0$ if $f(x)$ possesses an absolutely integrable bounded third derivative.

For $f(0)=f(\pi)=0$, $|f(x)|\leqq\bar{f}$, and $2(1-2s)k-h^2\leqq 0$, $0\leqq s\leqq 1$, then

$$|U(mh,nk)|\leqq C(h_1)\bar{f}\log 1/h.$$

Thus, the cumulative effect of all errors at preceding grid points is less than $\varepsilon Ctk^{-1}\bar{f}\log 1/h$, where ε is an upper bound for the error committed at one point.

G. W. Evans, II (Menlo Park, Calif.)

6796:

Uhlmann, W. Differenzenverfahren für die 1. Randwertaufgabe mit krummflächigen Rändern bei $\Delta u(x,y,z)=r(x,y,z,u)$. Z. Angew. Math. Mech. 38 (1958), 130–139. (English, French and Russian summaries)

Extending from two to three dimensions earlier results of J. Albrecht and W. Uhlmann [same Z. 37 (1957), 212–224; MR 19, 884] the author derives finite difference formulas for the equation $u_{xx}+u_{yy}+u_{zz}=r(x,y,z,u)$, which he considers in a finite region with curved boundary surfaces. The author is interested particularly in lattice points near the boundary and achieves there a local truncation error of the same order as for interior lattice points. The resulting difference methods for solving the boundary value problem (with u given on the boundary) are tested by numerical examples.

Walter Gautschi (Washington, D.C.)

6797:

Smirnov, S. V. Gronwall's fundamental theorem on nomogramizability. Dokl. Akad. Nauk SSSR 124 (1959), 34–37. (Russian)

6798:

Hovanskiĭ, G. S. Nomographic methods for an approximate representation of a function of one variable. Dokl. Akad. Nauk SSSR 121 (1958), 56–58. (Russian)

Two equations

$$(1)\quad F[v, f(u)\cos\alpha-g(u)\sin\alpha+A,$$
$$f(u)\sin\alpha+g(u)\cos\alpha+B]=0$$

$$(2)\quad G[f(u,v)\cos\alpha+g(u,v)\sin\alpha+A,$$
$$-f(u,v)\sin\alpha+g(u,v)\cos\alpha+B]=0$$

with two variables u and v and three parameters A, B and α are considered in this paper. The author shows how the nomograms which he constructed for these equations in another paper can be used for the solution of some particular problems on interpolation.

S. Kulik (Logan, Utah)

6799:

Foster, F. G. Upper percentage points of the generalized beta distribution. III. Biometrika 45 (1958), 492–503.

This paper is concerned with the solution of $P(\theta_{\max}\leqq x)=P$, where $\theta_{\max}$ is the maximum root of a certain determinantal equation. Tables for the case of two and three roots are given in parts I [Foster and Rees, Biometrika 44 (1957), 237–247; MR 19, 188] and II [Foster, Biometrika 44 (1957), 441–453; MR 19, 781]. The present table covers the case of four roots, and gives values of x to 4D, for $P=.80(.05).95$, $.99$, $\nu_1=5(2)195$, $\nu_2=4(1)11$, where ν_1 and ν_2 are the d.f. occurring in the determinantal equation.

I. Olkin (Stanford, Calif.)

COMPUTING MACHINES

See also 6359, 6957.

6800:

Wall, D. D. Multiplication time on the IBM 709. Math. Tables Aids Comput. 12 (1958), 217–218.

The multiplication time depends on the number of runs of zeros in the multiplier. To compute this, for the 2^n words of n bits each, it is found that: $R(n,l)$, number of runs of length l, $=(n-l+3)2^{n-l-1}$ for $n>l$, with $R(n,n)=2$; and $S(n,l)$, number of runs of length $\geqq l$, $=\sum_{x=1}^n R(n,x)=(n-l+2)2^{n-l}$.

From the known time for the various steps in a multiplication cycle it follows that the time for fixed point

multiplication (35 bits) is 193 microseconds and that for normalized floating point multiplication (27 bits) is 170 microseconds. *C. C. Gotlieb* (Toronto, Ont.)

6801:

Rappoport, M. I. A new program for difference calculations on perforated-card machines. Vyčisl. Mat. 3 (1958), 186–189. (Russian)

6802:

Nečiporuk, È. I. Scheme synthesis by linear transformations of variables. Dokl. Akad. Nauk SSSR **123** (1958), 610–612. (Russian)

6803:

Eldred, Richard D. Test routines based on symbolic logical statements. J. Assoc. Comput. Mach. **6** (1959), 33–36.

An approach to writing maintenance routines is developed, based on the logical diagrams of the system rather than on the machine instructions as a whole. The system is in use in connection with the Datamatic 1000 Central Processor. It has been formalized in a series of charts (not shown in detail) showing, among other things, the required machine instructions to produce the logical combinations desired, the effect of a failure on the operation of the machine and the program to be executed, and the detectability of the failure by built-in machine or program checking. *F. Edelman* (Princeton, N.J.)

6804:

Drozdov, B. M.; and Rappoport, M. G. Coding of operations on the electronic calculator ÈV80–3. Vyčisl. Mat. 2 (1957), 146–153. (1 insert) (Russian)

The EV80-3 is a card-programmed relay calculator (probably in operation at Leningrad, but this is not mentioned). A computational cycle on the machine occupies 110 milliseconds. The arithmetic unit adds, subtracts, multiplies, and divides, and rounds results to eight decimal digits. Numbers may be read from punched cards, stored in a memory unit of 40 decimal digits broken down into nine sub-registers, and later punched out.

A complete list of card instructions is given, including instructions to branch on the contents of various of the registers and to skip forward in time. A program for Milne's method of integrating an ordinary differential equation is also given. Most instructions operate on the 40 digits and replace them with the result.

J. W. Carr, III (Chapel Hill, N.C.)

6805:

Aleksidze, M. A. An algorithm for automation of the numerical solution of a plane Dirichlet problem for Laplace's equation. Dokl. Akad. Nauk SSSR (N.S.) **119** (1958), 847–850. (Russian)

A completely automatic "universal program" is described for solving Laplace's equation for the Dirichlet problem. The author describes a program for the high-speed electronic digital computer BESM (Institute of Precise Mechanics and Computing Techniques, Academy of Sciences, Moscow). The requirements on a problem for solution are that the regions being considered be bounded by a finite number of curves, given in parametric form, and satisfying a Lipschitz condition of degree 1. Regions may possess holes and "sharp necks".

The planar region is represented by a finite grid, which is mapped linearly into the internal memory unit of the computer. Information as to the geometry of the region is given by "local marking" of each piont in the grid, constructed by the "universal program". This program moves along the boundary of the region, mapping automatically from the curvilinear boundary for the original partial differential equation into a broken-lined boundary for the final partial difference equation, and noting information about each point in the lattice, for example, if it is an interior or boundary point, and what the surrounding configuration is for the latter. The algorithm for translating from the external language describing the configuration of the boundary into an internal code for each point in the finite grid is given in an incomplete form.

Finally the actual difference equations for the representation of Laplace's equation are given. In essence the program can then be said to examine a two-dimensional contour, map it into a finite-difference grid system, decide on the proper equation for computing the Liebmann iteration procedure at each point, and stop when the solution is obtained, all automatically. Input to the program consists of the boundary as described above, boundary values, and stopping criterion; Output consists of the finite-difference approximation to the solution of the original equation.

J. W. Carr, III (Chapel Hill, N.C.)

6806:

Zidarov, D. Solution expérimentale du problème de Dirichlet pour le demi-espace $Z>0$. C. R. Acad. Bulgare Sci. **11** (1958), 181–184. (Russian summary)

Dirichlet's problem for the half-space $z>0$ consists of determining the harmonic function $U(x, y, z)$ in this half-space such that the value of U is known at every point of the plane $z=0$. In this paper the author directs attention to a simple physical analogy for the problem. He notes that the desired function is proportional to the magnitude of the magnetic potential produced by the currents flowing in a suitable network disposed in the plane $z=0$. The apparatus which he uses to set up and to measure this potential, and which also includes the network just mentioned, is a well known experimental construction called the "spiral of Rogowsky". The author demonstrates by an example that sufficient accuracy is obtainable through the use of the device for its successful applications to problems in geophysics.

W. P. DeWitt (Washington, D.C.)

6807:

Zidarov, D. Solution expérimentale du problème de Neumann pour le semi-espace $Z>0$. C. R. Acad. Bulgare Sci. **11** (1958), 267–270. (Russian summary)

The problem of Neumann for the half-space $z>0$ consists of finding the function $U(x, y, z)$ in the region $z>0$ when one knows its vertical gradient $U_z(x, y, 0)$ at every point of the plane $z=0$. In this paper the author describes an analogue solution to the problem which employs as part of the method his analogue solution to a related problem (that of Dirichlet for $z>0$) which he reported in a previous memoir [#6806]. The author observes the following analogy: if one covers the plane $z=0$ with uniformly distributed currents such that the current intensity is proportional at any point to $U_z(x, y, 0)$, then the magnitude of the magnetic potential generated by these currents in the region $z>0$ will be proportional to $U_z(x, y, z)$. It will be noted that this function is the solution to Dirichlet's problem. That of Neumann is found by integrating $U_z(x, y, z)$ from infinity to z, a process that the author carries out by means of a further apparatus which he calls "Rogowsky's Shirt" and which is essentially a three-dimensional Rogowsky spiral. The

accuracy of the device is verified in the case of an example taken from applied geophysics.

W. P. DeWitt (Washington, D.C.)

6808:

Ramachandran, G. N.; and Krishnamurthy, E. V. "Līlāvati" - a new analogue computer for solving linear simultaneous equations and related problems. I. General principles and design of model I. Proc. Indian Acad. Sci. Sect. A. **48** (1958), 152–164. (1 plate)

In this machine, resistance, current and voltage are used to represent the quantities a_{ij}, x_j and b_i, respectively, in the equations

$$a_{ij}x_j=b_i.$$

Ohm's law is then applied to realise the equations, and a Gauss-Seidel iterative scheme is used for successive adjustment of the x_j. On this basis a design using only passive elements and a minimum of components is described. A three-equation prototype is illustrated, and the authors state that further improvements affecting design and operation are discussed in a following paper [not available to the reviewer].

J. G. L. Michel (Teddington)

MECHANICS OF PARTICLES AND SYSTEMS

6809:

Chambers, Ll. G. Solution of the Hund gravitational equations. Canad. J. Phys. **37** (1959), 433–437.

In gravitational fields as suggested by Hund:

$$\mathbf{F}=m\{\mathbf{f}+(\mathbf{v}\times\mathbf{g})\}$$

the vector fields **f** and **g** satisfy certain differential equations solutions of which are indicated in the present paper. As regards **f** the solution contains as a specialisation an **f** parallel to the velocity **v** and which, if there is no creation of matter, does not depend explicitly upon the density distribution. The solutions for **g** differ according to whether $\partial\mathbf{g}/\partial t$ is equal to zero or not.

If $\partial\mathbf{f}/\partial t$ and $\partial\mathbf{g}/\partial t$ both vanish there are two possibilities: 1. $\partial\mathbf{v}/\partial t=0$ ("steady universe"); 2. The density $\rho=0$ (universe devoid of matter). The simplified Hund equations in the latter case admit solutions corresponding to the geons of general relativity investigated by Wheeler [Phys. Rev. (2) **97** (1955), 511–536; MR **16**, 756].

E. B. Schieldrop (Oslo)

6810:

Irimiciuc, N. Sur le mouvement relatif du solide dont la masse est variable. Bul. Inst. Politehn. Iaşi (N.S.) **4(8)** (1958), 113–120. (Romanian. Russian and French summaries)

"Dans cette note sont exposés les théorèmes généraux et les équations de mouvement d'un solide dont la masse est variable par rapport à un repère, mobile à son tour envers un système absolu, dans les hypothèses que le centre de masse change de position dans le solide, et que le système d'axes de coordonnées, invariablement lié au solide, soit quelconque."

Du résumé de l'auteur

6811:

Irimiciuc, N. Sur la forme des équations de Lagrange pour des systèmes mécaniques de points, dont les masses sont variables. Bul. Inst. Politehn. Iaşi (N.S.) **4(8)** (1958), 99–102. (Romanian. Russian and French summaries)

6812:

★Боголюбов, Н. Н.; и Митропольский, Ю. А. Асимптотические методы в теории нелинейных колебаний. [Bogolyubov, N. N.; and Mitropol'skiĭ, Yu. A. Asymptotic methods in the theory of non-linear oscillations.] 2nd ed. revised and enlarged. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958. 408 pp. 17.55 rubles.

The first edition (1955) was reviewed in MR **17**, 368. The present edition contains a new chapter, 50 pages, on single-frequency oscillations in systems with many degrees of freedom. The applications of such oscillations to a study of forced vibrations are discussed. There is still no index.

J. L. Brenner (Menlo Park, Calif.)

6813:

Čertkov, R. I. Forced oscillations of a system actuated by an external alternating frequency power. Akad. Nauk Ukrain. RSR. Prikl. Meh. **4** (1958), 139–159. (Ukrainian. Russian and English summaries)

6814:

Faure, Robert. Sur la synchronisation des systèmes oscillants. Solutions voisines de points singuliers. C. R. Acad. Sci. Paris **247** (1958), 1087–1089.

6815:

Ziemba, Stefan. Vibrations of mechanical systems with one degree of freedom and generalized forces not depending in an explicit manner on time. Arch. Mech. Stos. **10** (1958), 649–669. (Polish and Russian summaries)

Consider the differential equation

$$q''+\varphi(q, q')+\psi(q)=0, \tag{1}$$

where $\psi(-q)=-\psi(q)$, $\psi(0)=0$, $q\psi(q)>0$ for $q\neq0$, $d\psi(q)/dq\geqq0$; $\varphi(q, 0)=0$, $\varphi(-q, q')=\varphi(q, q')$, $\varphi(q, -q')=-\varphi(q, q')$ $q'\varphi(q, q')>0$ for $q'\neq0$, $\partial\varphi/\partial q'\geqq0$. In a previous paper [same Arch. **10** (1958), 163–193; MR **20** #151], the author has discussed system (1) for the case where $\varphi(q, q')=\Phi(q')$ [Also, see ibid. **9** (1957), 487–504, 525–548; MR **19**, 899, 745]. The present paper is a continuation of his work on this equation. By using standard energy arguments, the qualitative behavior of the solutions of (1) is discussed and the trajectories in phase space are compared with a linearized version of (1). Also, he shows how to construct numerically the phase trajectories by a procedure called the δ-method [see, e.g., L.S. Jacobsen, J. Appl. Mech. **19** (1952), 543–553; MR **14**, 502].

J. K. Hale (Baltimore, Md.)

6816:

Manolov, S. A special case of the existence of small periodic motions of two penduli, subjected to uniform rotation. J. Appl. Math. Mech. **22** (1958), 192–197 (139–142 Prikl. Mat. Meh.).

Assume we have a Cartesian frame rotating about the vertical z-axis with an angular velocity ω. Now consider a sequence of bars of equal length and masses pinned end to end with the first one being attached to a point on the horizontal y-axis a distance R from the origin. The author considers the problem of the possible oscillation of such a sequence of bars when suspended under the action of a constant vertical gravitational field. It is shown that under appropriate initial conditions, which are physically realizable, small oscillations are possible about the vertical position.

H. M. Trent (Washington, D.C.)

6817:

Tipei, N.; et Guţă, C. Sur le mouvement de l'avion sur une trajectoire donnée. Acad. R. P. Romîne. Stud. Cerc. Mec. Apl. **9** (1958), 855–866. (Romanian. Russian and French summaries)

STATISTICAL THERMODYNAMICS AND MECHANICS

6818:

Salpeter, Edwin E. On Mayer's theory of cluster expansions. Ann. Physics **5** (1958), 183–223.

A topological rather than the usual combinatorial approach is applied to a classical macrocanonical ensemble of systems in which the constituent particles interact through two-body forces only. This method is applied to Mayer's irreducible cluster expansion, the fugacity and correlation functions. The method is applicable only when the cluster expansion converges rapidly, and thus can not be used for studying condensation phenomena. The case of Coulomb interactions is considered by looking at weak forces of long, though finite, range.

D. ter Haar (Oxford)

6819:

Maradudin, Alexei; and Weiss, George H. The disordered lattice problem: a review. J. Soc. Indust. Appl. Math. **6** (1958), 302–319.

A brief discussion of a regular lattice is first given, before considering in detail the formal treatments of Dyson and Schmidt for a disordered linear chain. A treatment of the three-dimensional disordered lattice by a perturbation procedure due to the authors is then dealt with, together with a brief account of a moment method due to Maradudin, Mazur, Montroll and Weiss.

S. Simons (London)

6820:

Mori, Hazime. Time-correlation functions in the statistical mechanics of transport processes. Phys. Rev. (2) **111** (1958), 694–706.

This paper is devoted to the quantum-mechanical calculation, for dilute gases, of the time correlations in terms of which the shear viscosity and thermal conductivity coefficients can be expressed (Kubo formula). Using random phases the time variation of Heisenberg operators is expressed in terms of the collision operator. The eigenfunctions and eigenvalues of the collision operator are obtained. The values of shear viscosity and thermal conductivity are deduced and are shown to reduce, for classical gases, to the values obtained in the Chapman-Enskog theory. The two last sections are devoted to a more thorough study of the time variation of Heisenberg operators, and show how the random phase assumption can be avoided in this problem.

L. Van Hove (Utrecht)

6821:

Kraichnan, Robert H. Statistical mechanics of coupled bosons in the Heisenberg representation. Phys. Rev. (2) **112** (1958), 1054–1055.

A brief outline is given of a method to deal with quantum many-body systems. It is based on the hierarchy of differential equations obeyed by the time dependent correlation functions (expectation values of products of creation and absorption operators). This system is closed by the usual procedure of neglecting all "irreducible" correlation effects beyond a certain order.

L. Van Hove (Utrecht)

6822:

Kraichnan, Robert H. Statistical mechanics of coupled particles in the Schrödinger representation. Phys. Rev. (2) **112** (1958), 1056–1057.

The method presented in the previous paper for the quantum many-body problem is here briefly reformulated in the Schrödinger representation.

L. Van Hove (Utrecht)

6823:

Mori, Hazime. Statistical-mechanical theory of transport in fluids. Phys. Rev. (2) **112** (1958), 1829–1842.

Here is an interesting treatment of irreversible processes in fluids, applicable to dense systems and presented in quantum-theoretical form. The fluid starts from a state of local thermal equilibrum, approaching the steady state. Irreversibility is introduced by a time averaging procedure applied to the density matrix. A special definition of the entropy is used, which was introduced in an earlier paper [J. Phys. Soc. Japan **11** (1956), 1029–1044; MR **18**, 611] by the author, and appears to be consistent with the macroscopic theory of irreversible processes. In the course of some very abbreviated calculations, formal expressions for the coefficients of viscosity, thermal conduction and diffusion are obtained which are similar to Kubo's for electrical conductivity. There are several approximations, of which some are adequately discussed.

H. S. Green (Adelaide)

6824:

★**Mazur, P. On statistical mechanics and electromagnetic properties of matter.** Advances in chemical physics, Vol. I, edited by I. Prigogine, pp. 309–360. Interscience Publishers, Inc., New York; Interscience Publishers, Ldt., London; 1958. xi+414 pp. $11.50.

The electromagnetic properties of bulk matter are derived from those of the constituent particles using standard statistical methods. The general theory is developed and applied to a derivation of Maxwell's equations (where the displacement vector and the magnetic field strength are defined in a way which differs slightly from the usual one), of the electrostrictive forces acting in a dielectric and to a discussion of scattering and refraction of light.

D. ter Haar (Oxford)

6825:

Münster, A. Zur Theorie der generalisierten Gesamtheiten. Molecular Phys. **2** (1959), 1–7.

Necessary and sufficient conditions on the density operator of a generalized statistical mechanical ensemble are explicitly exhibited which, when satisfied, lead to the equations of thermodynamics and the correct fluctuation formulae. The author points out that the pressure ensemble proposed by W. Byers Brown [Molecular Phys. **1** (1958), 68–82] satisfies only certain of these and, while it is consistent with thermodynamics, leads to wrong fluctuation formulae (in particular, those used in describing light scattering of a one-component system).

H. L. Frisch (Murray Hill, N.J.)

6826:

Devooght, J. New variational principle for transport theory. Phys. Rev. (2) **111** (1958), 665–667.

A general variational principle of Kahan, Rideau and Roussopoulos is applied to problems of transport theory, and, in particular, to the solution of the Milne problem. Advantages of the present approach are discussed.

S. Simons (London)

6827:

Diad'kin, I. G. On the solution of the kinetic equations of transport of neutrons or γ-ray quanta by the method of partial probabilities. Soviet Physics. JETP 34(7) (1958), 1039–1047 (1504–1517 Ž. Eksper. Teoret. Fiz.).

The problem of solving the transport equations for the slowing down and diffusion of neutrons and the analogous problem for γ-rays are reduced to quadratures by the well-known procedure of iteration in orders of scattering (i.e. expansion of the solution into a Neumann series). The case of energy-dependent cross sections permits simplification and is worked out in some detail. Extensive formulas are presented, which may be found convenient, but no numerical application is given. The reader may be warned that the main practical difficulty lies in the implementation of this formal procedure, because evaluation of 10–30 complicated terms of the Neumann series is usually required. An application under favorable conditions has been made by G. H. Peebles and M. S. Plesset [Phys. Rev. 81 (1951), 430–439; and unpublished reports].

U. Fano (Washington, D.C.)

ELASTICITY, PLASTICITY

See also 6614, 6910, 6911.

6828:

★Boley, Bruno A. Some observations on Saint-Venant's principle. Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 259–264. American Society of Mechanical Engineers, New York, 1958. xxvii+864 pp. $20.00.

This is primarily a summary of known facts concerning St. Venant's principle, leaning heavily on the theorem of Sternberg [Quart. Appl. Math. 11 (1954), 393–402; MR 15, 370] which made it precise. The author suggests that it may apply to parabolic as well as elliptic linear differential equations. Using Laplace's equations he obtains bounds on the error resulting from using the principle for a special class of regions.

J. L. Ericksen (Baltimore, Md.)

6829:

Deev, V. M. On the solution of the space problem of elasticity theory. Dopovidi Akad. Nauk Ukrain. RSR **1958**, 29–32. (Ukrainian. Russian and English summaries)

Attention is given to the use of harmonic and biharmonic functions in deriving general solutions of the equations of linear elasticity under equilibrium conditions. The present type of solution includes, as special cases, other published solutions and may also be reduced to the Boussinesq-Galerkin solution.

H. G. Hopkins (Fort Halstead)

6830:

Bosson, Geoffrey. Flexure of a slab on an elastic foundation. J. Proc. Roy. Soc. New South Wales **92** (1958), 36–42.

The reaction of the elastic foundation is postulated to depend on the applied load so as to satisfy the condition of 'contact' between the slab and the foundation. Thus, it is a generalization of the Westergaard hypothesis (reaction is proportional to the displacement).

R. M. Evan-Iwanowski (Syracuse, N.Y.)

6831:

Abramyan, B. L.; and Babloyan, A. A. Bending of thick circular slabs under axisymmetric loading. Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk **11** (1958), no. 4, 95–106. (Russian. Armenian summary)

6832:

Nowiński, Jerzy. An approximate theory of bending and torsion of straight solid bars. Rozprawy Inż. 4 (1956), 325–348. (Polish. Russian and English summaries)

A theory of bending, torsion and shear of straight bars of solid (circular, elliptic, rectangular and narrow symmetrical) cross-section is developed on the basis of two simplifying assumptions: (1) uniform warping of the cross-section along the axis of the bar, (2) transversal orthotropy of the material of such a type that cross-sections behave as rigid in their planes ($\nu=0$).

A. M. Freudenthal (New York, N.Y.)

6833:

Conway, H. D. Nonaxial bending of ring plates of varying thickness. J. Appl. Mech. **25** (1958), 386–388.

The author considers the bending problem for a circular ring plate of variable thickness which is subjected at its inner and outer edges to an arbitrary system of forces or moments. He considers only the case in which the thickness is assumed to be proportional to a power of r. He treats as an example a plate, clamped at its inner edge and free at its outer edge, which is subjected to a normal concentrated load on its outer edge. Other problems of this type have been considered by R. Gran Olssen [see, for instance, Ing.-Arch. 12 (1941), 123–132; MR 10, 85].

L. E. Payne (Newcastle-upon-Tyne)

6834:

★Heller, S. R., Jr.; Brock, J. S.; and Bart, R. The stresses around a rectangular opening with rounded corners in a uniformly loaded plate. Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 357–368. American Society of Mechanical Engineers, New York, 1958. xxvii+864 pp. $20.00.

This paper is concerned with the first boundary-value problem of plane isotropic elasticity for an infinite plate having a rectangular opening with rounded corners. The state of stress at infinity is biaxial and no special assumptions are made about the radius of curvature of corners and the "aspect" ratio (height of opening to width of opening). The solution is obtained by means of the method of complex function theory due to Kolosoff and Muskhelishvili. Numerical results are given for the case of uniform tension at infinity.

P. M. Naghdi (Berkeley, Calif.)

6835:

Bassali, W. A.; and Nassif, M. Transverse bending of infinite and semi-infinite thin elastic plates. III. Proc. Cambridge Philos Soc. **54** (1958), 288–299.

[For parts I and II, see Bassali, same Proc. 53 (1957), 248–255; Bull. Calcutta Math. Soc. 49 (1957), 119–127; MR 19, 904; 20#2917]. A thin plane elastic plate, elastically restrained at an inner circular boundary, is deflected by small transverse forces applied over a circular area. The restraint at the circular boundary includes, as two particular cases, a clamped and a simply supported boundary. The limiting case of a half-plane clamped along its straight edge is considered.

W. R. Dean (London)

6836:

Vorovič, I. I. Error of direct methods in the non-linear theory of shells. Dokl. Akad. Nauk SSSR 122 (1958), 196–199. (Russian)

The methods considered are those of Papkovich and Mushtari. Distance between the exact and approximate solutions is given by certain Hilbert space metrics. The order of magnitude of the error as the number of terms in the approximating function increases is determined for the two methods of approximation.

R. C. T. Smith (Armidale)

6837:

Fradlin, B. N.; and Šahnovs'kiĭ, S. M. On obtaining integro-differential equations for the equilibrium of gently inclined shells. Dopovidi Akad. Nauk Ukrain. RSR 1958, 381–385. (Ukrainian. Russian and English summaries)

The authors apply Kiltchevsky's method [Akad. Nauk Ukrain. RSR Zb. Prac' Inst. Mat. **1940**, no. 4, 83–149; no. 5, 73–97; **1941**, no. 6, 51–105; **1946**, no. 8, 97–110; MR **2**, 172; **3**, 31; **12**, 372] and reduce the title problem to a study of a system of functional equations with Green's tensor for shells. Using Kirchhoff-Love's methods the nuclei and the operators entering into the equations are constructed. One discusses an example for computing a hinge-supported, gently inclined shell of rectangular plan acted on by a continuously distributed load.

D. P. Rašković (Belgrade)

6838:

Remizova, N. I. Determination of elastic displacements in cylindrical shells by the integral equation method. Dopovidi Akad. Nauk Ukrain. RSR **1958**, 263–266. (Ukrainian. Russian and English summaries)

As in a paper of Fradlin and Šahnovskii, the author applies Kiltchevsky's method [Akad. Nauk Ukrain. RSR Zb. Prac' Inst. Mat. **1946**, no. 8, 97–110, MR **12**, 372] to the determination of displacements in cylindrical shells with a guide of arbitrary form. Starting with Kirchhoff's hypothesis, supposing that the boundary conditions represent a combination of rigid and hinged fastening of the shell ends, the author reduces the title problem to the solution of an integral equation of Fredholm's type of the second order with regular kernel. As an example the problem with radial force and hinged shell is treated.

D. P. Rašković (Belgrade)

6839:

Vorovič, I. I. On some direct methods in the non-linear theory of vibrations of curved shells. Izv. Akad. Nauk SSSR. Ser. Mat. 21 (1957), 747–784. (Russian)

The application of the Bubnov-Galerkin method to the approximate solution of non-linear dynamical problems in general is first discussed. Then the theory of vibrating thin shells is treated as an example.

R. C. T. Smith (Armidale)

6840:

Conway, H. D. Some special solutions for the flexural vibration of discs of varying thickness. Ing.-Arch. 26 (1958), 408–410.

The natural frequencies of axisymmetrical vibrations of discs whose flexural rigidity varies like r^m (r is the radius) are computed exactly for several values of m and corresponding values of Poisson's ratio. Numerical results are given for a disc with a clamped edge and are compared with the results for a disc with uniform thickness.

R. C. DiPrima (Troy, N.Y.)

6841:

★Dill, E. H.; and Pister, K. S. Vibration of rectangular plates and plate systems. Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 123–132. American Society of Mechanical Engineers, New York, 1958. xxvii+864 pp. $20.00.

A procedure for analyzing the steady-state vibration of single-span plates and continuous plates on non-deflecting supports is presented. The method is analogous to the slope-deflection method developed by Prager [Ing.-Arch. 1 (1930), 527–532] for the analysis of the vibration of continuous beams and frames. A seven-moment equation is derived for a plate continuous over non-deflecting supports; it expresses the continuity condition at a support and is used in the same manner as the three-moment equation in the theory of continuous beams; it may be considered as a generalization of the seven-moment equation derived by Kalmanok [Structural mechanics of plates, (Russian), Moscow, 1950] for the static deflection of continuous plates.

G. B. Warburton (Edinburgh)

6842:

★Huang, T. C. Effect of rotatory inertia and shear on the vibration of beams treated by the approximate methods of Ritz and Galerkin. Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 189–194. American Society of Mechanical Engineers, New York, 1958. xxvii+864 pp. $20.00.

A straightforward discussion of the use of the Ritz and Galerkin methods for the solution of beam problems incorporating the Rayleigh rotatory inertia and the Timoshenko shear deformation approximations. As an illustrative example, the case of a simply supported, freely vibrating beam is treated; the solution to this problem is immediately evident from an inspection of the differential equation for the beam motion.

M. Goland (San Antonio, Tex.)

6843:

Morley, L. S. D. The flexural vibrations of a cut thin ring. Quart. J. Mech. Appl. Math. 11 (1958), 491–497.

The paper deals with both symmetric and antisymmetric flexural vibrations of a cut thin ring, using the usual assumptions of linear elasticity theory. The characteristic equations are derived and solutions are given which are asymptotic with respect to the order of the frequency. Results are tabulated for the first ten modes; these indicate that the asymptotic formulas given are reasonably accurate even for the first mode, especially for symmetric vibrations. The author states that the method is easily extended to vibrations of circular arcs, but does not discuss this problem further.

W. E. Boyce (Troy, N.Y.)

6844:

Alumyaè, N. A. Critical pressure of a shell generated by ellipsoidal surface. Eesti NSV Tead. Akad. Toimetised. Tehn. Füüs.-Mat. Tead. Seer. 5 (1956), 175–190. (Russian. Estonian and English summaries)

The middle surface of the shell is a surface of revolution generated by a portion of an ellipse, the two ends are clamped and it is subjected to uniform external pressure. Finding the critical pressure corresponds to solving an eigenvalue problem for a pair of ordinary differential equations containing a large parameter. This is treated by

a method of asymptotic integration. Some numerical results are given for a special case.

R. C. T. Smith (Armidale)

6845:

Chakraborty, S. K. On disturbances produced by a time-periodic twist on the surface of a spheroidal cavity. Geofis. Pura Appl. 40 (1958), 15–18.

There is a prolate spheroidal-shaped cavity contained within an elastic medium otherwise of indefinite extent in all directions. The surface of the cavity is supposed subject to an harmonic shear stress distribution corresponding to an oscillatory torsional couple about the axis of the cavity. With use of prolate spheroidal co-ordinates, the single equation of motion is reduced through the method of separation of variables and hence the general solution of this equation is found. An explicit solution is then obtained for the periodic motion due to a particular type of applied shear stress distribution. It is stated that the corresponding problem for an oblate spheroidal-shaped cavity may be similarly solved. No numerical results are presented. *H. G. Hopkins* (Fort Halstead)

6846:

Sveklo, V. A. On the theory of impacts of cylinders. Soviet Physics. Dokl. **120**(3) (1958), 530–534 (47–50 Dokl. Akad. Nauk SSSR).

The problem considered is one in the linear theory of elasticity. The cylinders, with their generators parallel, are in a state of plane strain. The bounding curves of their right sections, before deformation, are represented by power series in the neighborhood of the area of contact. Wave propagation is taken into account, with the cylinders treated as half-spaces. The analysis results in formulas for duration of contact and maximum pressure, width of contact area and relative approach as functions of impact velocity and properties of the cylinders. An example is given of the impact of a circular cylinder on a concave parabolic cylindrical surface.

R. D. Mindlin (Katonah, N.Y.)

6847:

✱Davids, Norman. Stress waves of penetration in plates. Proceedings of the Third Congress on Theoretical and Applied Mechanics, Bangalore, December 24–27, 1957, pp. 35–48. Indian Society of Theoretical and Applied Mechanics, Indian Institute of Technology, Kharagpur, 1958. xi+362 pp.

The author considers the stress distribution in a perfectly elastic plate, of finite thickmess but infinite extent, due to a point load varying harmonically with time. The solution is expressed as a contour integral, and evaluated for the direct stress along the axis of impact.

J. W. Craggs (Providence, R.I.)

6848:

Bhatia, A. B. Scattering of high-frequency sound waves in polycrystalline materials. J. Acoust. Soc. Amer. 31 (1959), 16–23.

Le sujet traité est la diffusion des ondes élastiques dans un milieu isotrope, formé de grains dont la densité et les coefficients d'élasticité oscillent légèrement autour de leurs valeurs moyennes. L'intensité de la diffusion est mesurée par un coefficient de perte d'énergie σ. Si l'énergie, par volume unitaire, dans un train d'ondes planes se propageant suivant la direction Ox, est I_0 à l'origine et I à l'abscisse, $I=I_0\exp(-6x)$. L'auteur définit les ondes élastiques diffusées selon la méthode de Lord Rayleigh [Theory of sound, Vol. II, 2d ed., Dover Publications, New York, 1945; MR 7, 500; p. 149]. Il trouve que toute onde élastique, longitudinale ou transversale, donne toujours naissance, par sa diffusion, à des ondes longitudinales et à des ondes transversales. Il évalue les coefficients de perte σ, en supposant que les oscillations ont une longueur d'onde grande par rapport aux dimensions des grains. Les résultats obtenus sont en assez bon accord avec les données expérimentales.

Enfin, l'auteur examine la diffusion des ondes élastiques par les fluctuations locales de la densité qui résultent de l'agitation thermique; et il conclut que cette diffusion est seulement appréciable dans les gaz au voisinage du point critique. *J. Laval* (Paris)

6849:

Sherwood, J. W. C. Propagation in an infinite elastic plate. J. Acoust. Soc. Amer. **30** (1958), 979–984.

The author discusses the propagation of waves in an infinite elastic plate, extending the work of several previous writers to include the case of complex propagation constants. The associated vibrations are exponentially attenuated and have finite phase velocities. Solutions of the characteristic equation are presented in graphical form. *W. E. Boyce* (Troy, N.Y.)

6850:

Borodačov, M. M. Longitudinal vibrations of viscoelastic rods. Akad. Nauk Ukrain. RSR. Prikl. Meh. **4** (1958), 176–181. (Ukrainian. Russian and English summaries)

The well-known equations of one-dimensional longitudinal vibrations of simple linear visco-elastic rods are restated and the problem of impact (prescribed velocity) at one end of a finite Maxwell bar is solved by the use of Laplace transforms. *A. M. Freudenthal* (New York, N.Y.)

6851:

Terry, N. B. The behaviour of a vibrating viscoelastic cylinder. Proc. Phys. Soc. 71 (1958), 973–978.

Steady state oscillation of a rod of viscoelastic material is analysed. The theory applies to a rod in torsion, or in tension neglecting lateral effects. The four parameter viscoelastic model composed of a Maxwell model and a Kelvin (Voigt) model in series, termed a Burgers body, is studied. Steady state forced oscillation is analysed by standard methods associated with prescribed oscillatory force on one end. Properties of the resonance curve for motion of the free end are used to determine material constants. To find all four constants, asymptotic behaviour in creep is also utilized and, since the author states that the material constants may be frequency dependent, the reviewer feels that caution must be observed in using this analysis for measuring material constants. *E. H. Lee* (Providence, R.I.)

6852:

✱Sternberg, Eli. On transient thermal stresses in linear viscoelasticity. Proceedings of the Third U.S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 673–683. American Society of Mechanical Engineers, New York, 1958. xxvii+864 pp. $20.00.

The author discusses the interrelation between elastic and linear visco-elastic thermal stress problems and establishes the general equations for conditions of polar symmetry. These equations are applied to formulate the complete solutions for the Maxwell and Kelvin bodies

under both stationary and transient temperature-fields, including numerical evaluation of the results.

A. M. Freudenthal (New York, N.Y.)

6853:

Drummond, W. E. Multiple shock production. J. Appl. Phys. 28 (1957), 998–1001.

The pressure-volume relation for a metal under high pressure is concave upwards, apart from the influence of a phase change which can produce a discontinuity of gradient equivalent to a region of infinite curvature concave downwards. These properties lead to multiple shock production for loading waves, one on each side of the phase change condition, and a rarefaction shock on unloading at the phase change condition. Idealised initial motion problems are solved to illustrate the types of wave interaction which can arise, using the method of characteristics for waves in one dimension. A hydrodynamic type theory of metal deformation which neglects yield is assumed, and a third order approximation in shock strength is used. Qualitative agreement with experiment is achieved. *E. H. Lee* (Providence, R.I.)

6854:

Lianis, G.; and Ford, H. Plastic yielding of single notched bars due to bending. J. Mech. Phys. Solids 7 (1958), 1–21. (1 plate)

Slip line fields according to the theory of ideal plasticity in plane strain are constructed for slabs in pure bending with a notch on one face. Various symmetrical notch shapes are considered. These fields are used to determine upper bounds for the limit moment, since the condition of nonnegative plastic rate of working is established. Lower bounds are determined by discontinuous stress fields. Experiments give limit moments close to the upper bounds, and this is confirmed by the agreement of the deformation fields associated with the slip line fields and measured by means of scribed grids.

E. H. Lee (Providence, R.I.)

6855:

Schlechtweg, H. Zur ebenen Plastizität bei spannungabhängiger Kohäsion. Z. Angew. Math. Mech. 38 (1958), 139–148. (English, French and Russian summaries)

The plane plastic problem is discussed under the assumption of a conical yield surface which, in the plane problem, can be expressed as an enveloping curve consisting of two straight lines in the Mohr $\tau_n-\sigma_n$-plane: $\tau_n=k+\sigma_n \tan \rho$. For this yield condition the characteristics, which are real and identical with the glide-lines, are developed and the equilibrium equations integrated with their help. *A. M. Freudenthal* (New York, N.Y.)

6856:

Kuznetzov, A. I. The problem of torsion and plane strain of non-homogeneous plastic bodies. Arch. Mech. Stos. 10 (1958), 447–462. (Polish and Russian summaries)

A perfectly-plastic material whose yield stress k is a known function of the coordinates is considered. Methods used for the solution of problems with constant k are generalized to variable k. For torsion, a single differential equation can be obtained by two different methods. The angle ϕ between a principal direction and the x axis can be introduced to satisfy the yield condition identically, or a stress function can be used to satisfy the equilibrium equation identically. The resulting characteristics are the lines of maximum shear stress and are normal to the boundary. If k is constant, the characteristics are straight; for variable k they will generally be curved. An exception is the case where k is a function only of the normal distance from the boundary. A rectangular bar where $k=k_0 \exp[\alpha(x+a)]$ is solved as an example.

The author mentions that the elastic-plastic problem may be solved by a suitable modification of the Nadai sand hill-soap film analogy. Since for some functions k the plastic zone may start in the interior, it does not appear to this reviewer that the resulting solution is necessarily correct.

For plane strain, the mean normal stress σ and angle θ between a principal direction and the x axis are introduced to satisfy the yield condition identically. As is the case for constant k, the resulting differential equations are hyperbolic and have orthogonal characteristics in the directions of maximum shear. Thus, numerical methods evolved for constant k may be used for variable k as well. Examples considered include a circular hole in an infinite sheet under uniform pressure and tangential load, where $k=k(y)$, and a half space loaded by a uniform pressure over $a-1\leq x\leq 1$ finite region, where $k=k_0+cy$, $c\ll k_0$.

Also discussed are the velocity solutions for plane strain and a linear perturbation method when $k=k_0+k_1(x, y)$, where $|k_1|\ll k_0$. *P. G. Hodge, Jr.* (Chicago, Ill.)

6857:

Madejski, Jan. The buckling of a prismatic bar as a problem of dynamical theory of plasticity. Rozprawy Inż. 4 (1956), 351–366. (Polish. Russian and English summaries)

A discussion of the time-effect in elastic-plastic buckling is based on a one-dimensional so-called "dynamic plasticity" equation which is, in fact, the equation of a Maxwell strut with a "yield-limit". It is obvious that if time-sensitive material behavior is assumed, a time-sensitive response will result. It is, however, unlikely that the behavior of a metal at or close to room-temperature can be represented by the introduction of an elastic-visco-plastic model. *A. M. Freudenthal* (New York, N.Y.)

6858:

Madejski, Jan. The dynamical theory of plasticity as a link between the theory of elasticity and the theory of plasticity. Rozprawy Inż. 6 (1958), 467–481. (Polish. Russian and English summaries)

The one-dimensional beam equation is solved under the assumption that the material is elastic for stresses below the yield limit and is a Maxwell body with constant relaxation time for stresses exceeding it. For some unexplained reason, the above assumption is called a "dynamical theory of plasticity"; it is neither "dynamical" nor, in fact, a theory of "plasticity". The paper gives a one-dimensional application of the equations of the Bingham body with an elastic term.

A. M. Freudenthal (New York, N.Y.)

6859:

Shield, Rich. Thorpe; and Ziegler, Hans. On Prager's hardening rule. Z. Angew. Math. Phys. 9a (1958), 260–276. (German summary)

Prager's hardening rule is based on a kinematic model in which the yield surface is visualized as a rigid frame in stress space. When the material is plastic, the stress point is in contact with this frame and moves it in translation only.

Logically, this rule must be applied in a 9-dimensional stress space. If some stress components are identically zero (e.g. plane stress, torsion, etc.), then at any given

instant it is sufficient to consider a lower-dimensioned subspace. However, unless the yield frame is normal to this subspace, the motion of the frame will have a component out of the subspace. It follows that the kinematic model cannot be directly applied to the subspace.

The present paper considers this problem in some detail. A precise mathematical formulation of the kinematic model is given in 9-space and particularized to various subspaces. In some cases, the model of the frame as a rigid frame may be maintained provided that a suitable modification is made for its mode of motion. In other cases, the size or shape of the intersection of the frame and the subspace may change as hardening progresses. Which phenonemon occurs is found to depend not only on the particular subspace being considered, but also on the form of the initial yield condition.

P. G. Hodge, Jr. (Chicago, Ill.)

6860:

Naghdi, P. M. On plane stress solution of an elastic, perfectly plastic wedge. J. Appl. Mech. **25** (1958), 407–410.

The wedge is subject to uniform pressure along one edge. Using Tresca's yield criterion, the author shows that yielding commences at both edges simultaneously. In the plastic region that forms on the edge subjected to pressure, the yield point is on the side of the Tresca hexagon, in that on the other edge it is at a vertex of the hexagon. Although this result is different from that for the corresponding problem in plane strain, components of stress in the plane are identical in the two cases and the equations for the displacements have the same form but different coefficients.

D. R. Bland (Manchester)

6861:

Das, Sisir Chandra. On the general plane problem of plasticity and its geophysical significance. Canad. J. Phys. 37 (1959), 63–74.

The main part of this paper is an acknowledged restatement of results obtained by three previous authors: Neuber, Sauer, and Sokolovsky (references are given in the paper). The author suggests that these methods may be useful in explaining observed displacements in earthquakes centered in the orogenic belt. He points out that the application will be difficult because the appropriate form of the yield condition is unknown and boundary conditions are not well defined.

P. G. Hodge, Jr. (Chicago, Ill.)

6862:

Ivlev, D. D. A class of solutions of the general equations of the theory of ideal plasticity. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1958**, no. 11, 107–109. (Russian)

A plastic mass obeying the yield and flow rule of von Mises is pressed between rigid platens. A solution is obtained for the initial motion problem in which the strain rates are constant in magnitude and direction. The plane strain solution due to Prandtl and the axisymmetric solution due to Hill emerge as special cases.

R. M. Haythornthwaite (Providence, R.I.)

6863:

Hodge, P. G., Jr. A general theory of piecewise linear plasticity based on maximum shear. J. Mech. Phys. Solids **5** (1957), 242–260.

The paper is concerned with the formulation of strain-hardening behavior of plastic solids. The familiar concepts of "yield surface" and "flow rule (incorporating the normality condition)" are retained, and the outstanding problem of the theory of plasticity, i.e., the dependence of the shape, size and position of the yield surface on the strain history, is considered. For any given class of materials extensive experimental evidence is needed for determining the strain-history dependence of the yield surface. However, as the author points out, speculations on mathematically consistent and convenient theories which retain the broad physical facts known about the strain-history dependence of plastic solids are also of great interest. The particular theory introduced by the author is based on the concept of maximum shear stress, predicts a Bauschinger effect and involves five material constants. The formulation is carried out in three-dimensional terms. The particular case of plane stress is also considered. The theory is compared with various previously introduced theories; Prager's rule of hardening [Proc. Inst. Mech. Engrs. **169** (1955), 41–57; MR **17**, 558] appears to be a special case of the present theory. It is further shown that, for certain types of loading, the incremental hardening rule introduced by the author may be explicitly integrated and that the classical principles of minimum potential energy and complementary energy are valid for this integrated law.

E. T. Onat (Providence, R.I.)

6864:

Romiti, Ario. Sull'equilibrio limite dei materiali pesanti dotati di coesione ed attrito interno. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **23** (1957), 400–408.

This paper considers the plane deformation of a heavy soil satisfying Coulomb's law of failure. The author finds the characteristics of stress and the relationships holding along them, and he classifies the various Cauchy problems which arise from different boundary conditions. Finally, the solution for a soil of infinite depth contained between two parallel walls is given. *D. R. Bland* (Manchester)

6865:

★Hanson, K. L.; and Horvay, G. Thermal stresses in a sector prism. Proceedings of the Third U. S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 347–356. American Society of Mechanical Engineers, New York, 1958. xxvii+864 pp. $20.00.

A sector of a heat-generating circular cylinder is considered in order to answer the question whether the stress can be reduced by sectioning the cylinder into sectors. The problem of the 60° sector is decomposed into the following stages. (1) Conventional plane strain, using two stress functions $\Phi_0+\Phi_I$ which secure the equilibrium of the sector and compensate for the force and the moment of the tractions on each individual edge of the boundary. (2) Compensation of self-equilibrating residuals, which are alarmingly large, on the edges. This follows by means of three correcting stress functions, one, Φ_{II}, which compensates for the hoop stress along the straight edges, and two, Φ_{III_r} and Φ_{III_θ}, which compensate for the residuals of tangential and radial stress along the arc. The compensation is done by means of a variational procedure using hypergeometric polynomials in r for Φ_{II} and in θ for Φ_{III} (r, θ plane polar coordinates). (3) The final stage consists in freeing the prism from the rigid end constraints.

Several graphs illustrate the distribution of stress.

J. Nowinski (Madison, Wis.)

6866:

Kovalenko, A. D. Some problems of thermo-elasticity in connection with temperature stresses in turbine rotors. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1958**, no. 10, 68–76. (Russian)

The following problems have been solved for a rotor disk with thickness h. (1) Plane stress and bending for variable h and cyclically symmetrical temperature field, assuming constant Young's modulus E across the thickness. This supplies two similar fourth order partial differential equations for the stress function and the deflection. (2) Complex bending for variable E across the thickness and spatial axially symmetrical temperature field. The system of equations is derived from the theory of shallow shells of revolution. In both cases the solution involves hypergeometric functions. (3) Spatial axially symmetrical thermal state for particular forms of variability of E, extending Lurie's method.

Numerical results are obtained by means of the differential analyzer "Integral" in Kiev. Experimental data show good agreement with the theoretical predictions.

J. Nowinski (Madison, Wis.)

STRUCTURE OF MATTER

6867:

Sewell, G. L. Electrons in polar crystals. Phil. Mag. (8) 3 (1958), 1361–1380.

The mobility of an electron in a solid depends on the interaction between the electron and the lattice particles. It has been found experimentally that in some ionic crystals the electronic mobility is exceptionally low. The author extends an earlier theory of this effect given by H. Fröhlich [Advances in Physics **3** (1954), 325–361.] It is assumed that the electron can be trapped on a positive ion, or on a set of equivalent positive ions near a defect center, where it produces a local distortion of the positions of the surrounding ions. This results in a tight coupling with the lattice vibrations, which reduces the electronic mobility. The theory is developed mathematically by calculation of the quantum mechanical interaction between the electron and the lattice oscillations. It is found that the electron mobility decreases with increasing temperature, owing to the effect of random thermal motions of the ions in opposing changes in their mean positions. *E. L. Hill* (Minneapolis, Minn.)

FLUID MECHANICS, ACOUSTICS

See also 6734, 6735.

6868:

Moiseev, N. N.; and Ter-Krikorov, A. M. The non-uniqueness of the solution to the under-water wing problem. Dokl. Akad. Nauk SSSR (N.S.) **119** (1958), 899–902. (Russian)

A wing of unspecified shape is submerged in a uniform flow of water of constant depth. The planar problem of determining the free surface contours in the resulting flow is investigated for Froude numbers well above and well below one by finding approximate solutions of a non-linear differential equation describing this surface. A separate analysis is then made to show that these solutions are of quite different nature when the Froude number is very close to one. Approximate expressions for the latter solutions are obtained and a sketch of the free surface contours is given.

J. F. Heyda (Cincinnati, Ohio)

6869:

Mimura, Yōichi. The flow with wake past an oblique plate. J. Phys. Soc. Japan 13 (1958), 1048–1055.

The theory devised by A. Roshko [N.A.C.A. Tech. Note no. **TN3168** (1954); MR **16**, 188] for the case of plane flow impinging normally on a flat plate is extended here to the flow about oblique plates at angle of incidence α. The results involve elliptic integrals. Numerical results have been calculated for a range of α between 15° and 70°. Detailed comparisons with experimental data are carried out, and the probable reasons for discrepancies at the lower values of α are discussed.

W. R. Sears (Ithaca, N.Y.)

6870:

Zwick, S. A. Behavior of small permanent gas bubbles in a liquid. I. Isolated bubbles. J. Math. Phys. **37** (1958), 246–268.

The motions of an isolated gas bubble in a nearly inviscid liquid are studied. The bubble is taken to be so small that surface tension will give it a spherical shape, and the temperature is assumed to be uniform except for variations near the bubble due to changes in its size. An approximate theory based on potential flow is developed. Radial motions caused by small oscillations in the liquid pressure are treated, and translations resulting from buoyancy due to gravity are discussed. Expressions are obtained for the pressure field under these circumstances.

P. R. Garabedian (Stanford, Calif.)

6871:

Ter-Krikorov, A. M. The nonlinear problem of the theory of a submerged airfoil. Soviet Physics. Dokl. **119** (3) (1958), 255–258 (1115–1117 Dokl. Akad. Nauk SSSR).

The problem considered is that of determining the complex potential for the motion of a two-dimensional aerofoil in an incompressible fluid bounded by a horizontal, solid boundary and a free surface. The method employs various complex transformations and the solution is expressed as a pair of non-linear integral equations. No expressions for the forces acting on the aerofoil are given and the method seems to be of theoretical interest only. *G. N. Lance* (Southampton)

6872:

Šerenkov, I. A. On a plane problem of spreading of a streaming incompressible fluid. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1958**, no. 1, 72–78. (Russian)

A layer of incompressible fluid with a free surface flows over a plane; in the supercritical or streaming motion the approximate equations of motion are hyperbolic, and the numerical method of characteristics has previously been applied. Here the case of a uniform stream emerging from between two parallel vertical walls is attacked analytically. Centered simple waves spring from each corner and subsequently interact; both regions are analyzed. Finally, the centered wave is abandoned in favor of a curved streamline from each corner, in accord with experimental observation. *M. D. Van Dyke* (Paris)

6873:

★Phillips, Owen M. Wave generation by turbulent wind over a finite fetch. Proceedings of the Third U.S.

National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 785–789. American Society of Mechanical Engineers, New York, 1958. xxvii+864 pp. $20.00.

When a turbulent wind blows across the surface of an inviscid liquid (water), waves are generated by the interaction of the convected pressure fluctuations and the free surface. In a previous paper [J. Fluid Mech. 2 (1957), 417–445; MR 19, 488] the author proposed a linear theory in which the wind structure was assumed independent of the wave motion in the water, and treated the initial development of waves on an unbounded ocean. He found that the rate of growth of each Fourier component of the wave field is greatest when the convection velocity of the component of the pressure fluctuations is equal to the velocity of free surface waves of the same wave-number. In the present paper the growth of waves over a finite fetch is studied, where the wind is assumed to blow off a straight shore on to a semi-infinite ocean. The wave motion is statistically stationary in time but develops with increasing distance from the shore. An expression is derived for the root-mean-square surface displacement; this is found to vary asymptotically as the square root of the distance from the shore at large distances from the shore. (The validity of linearization is assumed.)

F. Ursell (Cambridge, England)

6874:

Legen'kov, A. P. Plane free oscillations of an ideal homogeneous liquid in an infinite channel of variable cross-section. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1958**, 989–994. (Russian)

The paper concerns the classical problem of long waves in canals [H. Lamb, Hydrodynamics, 6th ed., Cambridge Univ. Press, 1932; pp. 254–278]. Solutions are obtained for canals of (a) constant depth and linearly varying width, (b) constant width and linearly varying depth, (c) linearly varying width and depth.

W. Kaplan (Ann Arbor, Mich.)

6875:

Kaplan, Paul. Comments on the paper: Waves produced by a pulsating source travelling beneath a free surface. Quart. Appl. Math. 16 (1958), 439–440.

The paper referred to in the title was by H. S. Tan [same Quart. **15** (1957), 249–255; MR **19**, 704], who treated the plane problem and found that there were 4 harmonic wave trains on the downstream side of the source when $0<\omega c/g<\frac{1}{4}$ (ω=circular frequency of pulsation, c=velocity of source). This result differs from the result obtained by E. Becker [Ing.-Arch. **24**, (1956), 69–76], P. Kaplan [Proc. Fifth Midwest Conference on Fluid Mechanics, 1957, pp. 319–329, Univ. of Michigan Press, Ann Arbor, Mich., 1957; MR **20** #539], and T. Y. Wu [Cal. Tech. Report 85-3, 1957], who agree that there are 3 harmonic wave trains on the downstream side, and 1 on the upstream side. In the present note an error in Tan's work is pointed out and corrected. The appropriate radiation condition at infinity can be found either by use of the 'Rayleigh viscosity' or, more convincingly, from the limit to which an unsteady wave pattern tends after a long time.

F. Ursell (Cambridge, England)

6876:

Cumberbatch, E. Two-dimensional planing at high Froude number. J. Fluid Mech. 4 (1958), 466–478.

The solution begins with Lamb's expression [Hydrodynamics, 6th ed., University Press, Cambridge, 1932; §§ 242–244] for the potential due to a pressure distribution at the water surface. This yields an integral equation for the pressure distribution on the planing body for given body slope. After a transformation of variables the integral is expanded in negative powers of K, where $K^{-\frac{1}{2}}$ is the Froude number. For $K=0$ the solution given by H. Wagner [Z. Angew. Math. Mech. 12 (1932), 193–215] is obtained; for the region below the surface it is the same as the solution for a thin airfoil. This can be used to begin an iteration procedure to calculate the terms of successively higher powers of K. Terms of order K^2 in the pressure on the planing body are worked out here. Lift and drag coefficients are calculated, and functions involved in their numerical determination are tabulated; viz.

$$\int_0^\phi \ln\tan\tfrac{1}{2}\theta d\theta \text{ and } \int_0^\phi \ln^2\tan\tfrac{1}{2}\theta\, d\theta \text{ for } 0\leqq\phi\leqq\pi/2,$$

$$\int_0^x (t^2-1)^{-1}\ln t\,dt \text{ and } \int_0^x (t^2-1)^{-1}\ln^2 t\,dt \text{ for } 0\leqq x\leqq 1.$$

The so-called splash drag identified by Wagner is determined, in particular; also the wave drag. Next, flat and parabolic planing surfaces are considered as special cases. It is noted that a flow without splash drag can be obtained at any given large Froude number by combining the flows due to these special cases. At these large Froude numbers the splash drag is by far the bigger part of the drag.

W. R. Sears (Ithaca, N.Y.)

6877:

Khamrui, S. R. On the flow of a viscous liquid through a tube of elliptic section under the influence of a periodic pressure gradient. Bull. Calcutta Math. Soc. 49 (1957), 57–60.

6878:

Khamrui, S. R. On the flow of a viscous liquid through a tube of elliptic section under exponential pressure gradient. Bull. Calcutta Math. Soc. 49 (1957), 147–152.

6879:

Khamrui, S. R. On the slow steady motion of an infinite viscous liquid due to the rotation of a cylinder. Bull. Calcutta Math. Soc. 49 (1957), 61–66.

6880:

Emersleben, Otto. Über die Parallelströmung zäher Flüssigkeiten zwischen koaxialen Zylindern im Grenzfall, dass das innere Rohr verschwindet. Z. Angew. Math. Mech. 38 (1958), 466–472. (English, French and Russian summaries)

6881:

Kiselev, A. A.; and Ladyženskaya, O. A. On the existence and uniqueness of the solution of the nonstationary problem for a viscous, incompressible fluid. Izv. Akad. Nauk SSSR. Ser. Mat. 21 (1957), 655–680. (Russian)

The authors study the initial and boundary value problem

(1) $\mathbf{v}|_{s=0}=0,\ \mathbf{v}|_{t=0}=\mathbf{a}(x);\ \mathbf{v}(x,t)=(v_1,v_2,v_3)$

in a bounded three dimensional region Ω with boundary S, for the Navier-Stokes partial differential equations

(2) $\dfrac{\partial\mathbf{v}}{\partial t}-\nu\Delta\mathbf{v}+\mathbf{v}\cdot\nabla\mathbf{v}+\nabla p=\mathbf{f}(x,t),\ \operatorname{div}\mathbf{v}=0$

and for the system

(3) $\dfrac{\partial\mathbf{v}}{\partial t}-\nu\Delta\mathbf{v}+\mathbf{v}\cdot\nabla\mathbf{v}=\mathbf{f}(x,t).$

Under an assumption that suitable square integrals of the initial vector $\mathbf{a}(x)$ and of its derivatives up to second order are sufficiently small (depending on Ω and on ν), the authors prove the existence of a generalized solution of (1), (2), having strong derivatives of the form $\mathbf{u}_{x_i}$, $\mathbf{u}_{tx}$, in the sense of an L_2 norm in a cylindrical region $\Omega\times[0\leq t\leq T]$. In case $\mathbf{f}(x, t)=0$, the existence is demonstrated for the entire semi-infinite cylinder $t\geq 0$ with Ω as base. Uniqueness is proved for an apparently broader class of generalized solutions. Proofs make use of L_4 estimates for general classes of functions due to Sobolev.

The above results overlap earlier work of E. Hopf [Math. Nachr. **4** (1950), 213–231; MR **14**, 327], who proved, without restriction on the initial data, the existence for all time of a generalized solution of the homogeneous problem having strong derivatives of first order.

The authors discuss also the solutions of (1), (3). It has been pointed out by L. Nirenberg that the proof of existence given in this case is incorrect, for the conclusion of compactness in L_2 norm which the authors draw from relation (49) p. 675 admits a simple counterexample.

R. Finn (Stanford, Calif.)

6882:

Dolidze, D. E. Unsteady flow of viscous fluid between parallel porous walls. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 380–383. (Russian)

Berman [J. Appl. Phys. **24** (1953), 1232–1235; MR **15**, 573] investigated steady plane flow of a viscous incompressible fluid between parallel walls of uniform porosity. Here, taking that flow as the initial state, the unsteady case is studied when the velocity through the walls varies with time. Various transformations involving solutions of the heat equation lead to a system of nonlinear integral equations, which is solved by iteration for small porosity. *M. D. Van Dyke* (Paris)

6883:

Kulonen, G. A. Interaction of a shock wave with the boundary layer of the leading edge of a flat plate at high supersonic speeds with radiation. Vestnik Leningrad. Univ. **13** (1958), no. 7, 172–188. (Russian. English summary)

The theory of strong interaction between the hypersonic boundary layer and resulting shock wave is extended to include radiation from the plate according to the Stefan-Boltzmann law (radiation of the gas itself being neglected). The boundary layer is assumed to extend to the shock wave (this model, though incorrect, is known to give qualitatively correct results). An integral method is used, the velocity profile and surface temperature being approximated respectively by 4- and 2-term polynomials. Numerical results are given for Mach numbers of 10 and 15 at an altitude of 20 kilometers.

M. D. Van Dyke (Paris)

6884:

Šidlovskiĭ, V. P. On the role of slip in the flow of a viscous gas past a plane semi-infinite plate. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1958**, no. 9, 83–90. (Russian)

A region near the leading edge is considered where the ratio of mean free path to boundary layer thickness is small but appreciable, its square being negligible. Then a first effect of slip can be found by solving the ordinary boundary layer equations with modified conditions at the surface. With Prandtl number of unity and viscosity proportional to temperature, the solution is found for a flat plate that is insulated or at constant temperature. As a matter of interest, the solutions are carried as far as terms in the square of mean free path, although this exceeds the validity of the boundary layer equations.

M. D. Van Dyke (Paris)

6885:

Dorfman, A. Sh.; Pol'skiĭ, N. I.; and Romanenko, P. N. Self-similar solutions of the laminar boundary layer equations for a compressible fluid including heat transfer. J. Appl. Math. Mech. **22** (1958), 375–382 (274–279 Prikl. Mat. Meh.).

An enumeration is given of all combinations of external velocity distribution, surface temperature distribution and Prandtl number for which the boundary-layer equations in plane flow can be reduced to ordinary differential equations. No consideration is given to the questions of existence and uniqueness, which are known to be difficult even for the incompressible Falkner-Skan solutions. *M. D. Van Dyke* (Paris)

6886:

Meksyn, D. The boundary layer equations of compressible flow. Separation. Z. Angew. Math. Mech. **38** (1958), 372–379. (German, French and Russian summaries)

The Illingworth-Stewartson transformation is combined with a method previously given by the author [D. Meksyn, Proc. Roy. Soc. London Ser. A **237** (1956), 543–559; MR **18**, 693] for incompressible boundary layers and applied to find the position of separation in a shock-wave interaction with a laminar boundary layer, using the experimental pressure gradient. The author is not wholly satisfied with his results.

K. Stewartson (Durham)

6887:

Curle, N. The steady compressible laminar boundary layer, with arbitrary pressure gradient and uniform wall temperature. Proc. Roy. Soc. London. Ser. A **249** (1959), 206–224.

This paper describes an approximate method of calculating the position of separation in a compressible boundary layer when there is heat transfer from the wall. It is assumed that the viscosity varies as the absolute temperature T and that the Prandtl number is unity. T is assumed to be a quadratic function of the velocity u whose coefficients are determined from the conditions at the wall and at the outer edge of the boundary layer. With this form for T the momentum equation is solved using a generalisation of Thwaites' method for incompressible flow, the universal functions being effectively the same.

The method is applied to a boundary layer in which the pressure is a linear function of distance from the leading edge, and the agreement with the exact solution is encouraging. It is also applied, however, to two boundary layers with a linear main stream velocity, and the agreement with the exact solutions is not so good, errors of about 50 per cent being reported in the position of separation.

{The poor agreement in the last two examples may be due to the choice of universal functions. It may be better to take a leaf out of Thwaites' book and use all the exact solutions which have been computed to obtain more widely applicable universal functions.}

K. Stewartson (Durham)

6888:

Hassan, H. A. A new solution to the laminar boundary-layer equations. J. Aero./Space Sci. **26** (1959), 189–190.

6889:

Thompson, Philip Duncan. A heuristic theory of large-scale turbulence and long-period velocity variations in barotropic flow. Tellus 9 (1957), 69–91.

6890:

Batchelor, G. K. Small-scale variation of convected quantities like temperature in turbulent fluid. I. General discussion and the case of small conductivity. J. Fluid Mech. 5 (1959), 113–133.

This paper is concerned with the small scale structure of a dynamically passive, conserved scalar quantity θ in a turbulent fluid. An illuminating discussion is given of previous contributions pointing out an apparent conflict between the results of Obuhov and Corrsin on the one hand and Batchelor on the other. A close examination of the two arguments reveals that each is relevant under different circumstances. If the kinematic viscosity is much less than the diffusivity κ, the turbulent energy spectrum extends to higher wave-numbers than does the θ-spectrum, which has a conduction cut-off near the wave-number $n=(\varepsilon/\kappa^3)^{1/4}$, where ε is the energy dissipation per unit mass. This is the result suggested by Obuhov [Izv. Akad. Nauk, SSSR. Geograf. Geofiz. 13 (1949), 58–69; MR 11, 552]. On the other hand, if $\nu\gg\kappa$ the θ-spectrum extends to higher wave-numbers than the energy-spectrum, cutting off near $n=(\varepsilon/\nu\kappa^2)^{1/2}$, which is Batchelor's earlier result [Proc. Roy. Soc. London Ser. A 213 (1952), 349–366; MR 14, 698]. These predictions are all based on simple extensions of the Kolmogoroff postulates.

The case $\nu\gg\kappa$ is examined in detail, and it is shown that the θ-spectrum is proportional to n^{-1} for $(\varepsilon/\nu^3)^{1/4}\ll n\ll(\varepsilon/\nu\kappa^2)^{1/2}$ and cuts off exponentially at the upper end. This paper resolves a difficulty of some years' standing, and its clarity and ease of presentation make it particularly valuable. *O. M. Phillips* (Baltimore, Md.)

6891:

Batchelor, G. K.; Howells, I. D.; and Townsend, A. A. Small-scale variation of convected quantities like temperature in turbulent fluid. II. The case of large conductivity. J. Fluid Mech. 5 (1959), 134–139.

This contribution is concerned with the form of the conduction cut-off when $\kappa\gg\nu$ and the θ-spectrum terminates at lower wave-numbers than does the energy spectrum. Making use of an assumption concerning the convective stretching of the θ-field it is shown that, when $(\varepsilon/\kappa^3)^{1/4}\ll n\ll(\varepsilon/\nu^3)^{1/4}$, the θ-spectrum decreases as $(n)^{-17/3}$. An interesting point which is demonstrated explicitly in the discussion is that the action of convection at these wave-numbers is directly influenced by the conduction and cannot be considered separately.

O. M. Phillips (Baltimore, Md.)

6892:

Shigemitsu, Yutaka. Non-similarity theory of decaying turbulence. J. Phys. Soc. Japan 14 (1959), 91–103.

L'auteur a déjà considéré [même J. 10 (1955), 472–482, 890–902, 1077–1087; MR 17, 312, 680] la turbulence derrière un obstacle comme une superposition de tourbillons de dimensions décroissantes, les tourbillons de dimension $m+1$ empruntant leur énergie à ceux de dimension m.

Deux hypothèses simples conduisent en première approximation d'attacher aux probabilités de passage, dans cette cascade de tourbillons, des distributions de Poisson. En faisant intervenir une similitude locale pour chaque classe de tourbillons, on obtient des expressions pour l'énergie turbulente, les corrélations doubles et les corrélations triples en turbulence isotrope. Des valeurs numériques sont calculées et confrontées avec diverses valeurs expérimentales. *J. Bass* (Paris)

6893:

Rose, Peter H.; Probstein, Ronald F.; and Adams, Mac C. Turbulent heat transfer through a highly cooled, partially dissociated boundary layer. J. Aero./Space Sci. 25 (1958), 751–760.

The problem of heat transfer through a turbulent boundary layer is attacked by considering the equations of continuity, motion and energy, and assuming (i) negligible density fluctuations, (ii) transport of momentum and enthalpy (including dissociation energy) to be described by coefficients of eddy transfer that are the same, i.e., the turbulent Prandtl number is unity. Then the equations have the same form as for non-dissociating gases and variations of velocity and enthalpy are linearly related if there is no pressure gradient. For highly cooled layers, it is shown that the variations of enthalpy and density in the outer, turbulent part of the layer are small and it is suggested that the incompressible coefficient of skin-friction may be used. If the laminar Lewis number is different from unity, it is suggested that a correction based on the effect of this on transfer through the laminar sub-layer should be used. Experimental work in a shock-tube is described which supports this analysis.

A. A. Townsend (Cambridge, England)

6894:

Oswatitsch, K.; and Teipel, I. Verträglichkeitsbedingungen für instationäre Strömung. Z. Angew. Math. Mech. 38 (1958), 73–74.

The authors consider the equations of one dimensional flow,

$$(1)\quad \frac{d\rho}{dt}+\rho\frac{\partial W}{\partial x}+\frac{\rho}{f}\frac{df}{dt}=0,\quad \frac{dW}{dt}+\frac{1}{\rho}\frac{\partial p}{\partial x}=0,\quad \frac{ds}{dt}=0,$$

in the cases that the cross section f is a function only of x or only of t. They show that by introduction of characteristic coordinates ξ, η, (1) can be transformed, in the former case, to

$$\left(\frac{\partial(\rho Wf)}{\partial t}\right)_\xi-\frac{\rho}{c}f\left(\frac{\partial(\frac{1}{2}W^2+i)}{\partial t}\right)_\xi=0,$$

$$\left(\frac{\partial(\rho Wf)}{\partial t}\right)_\eta+\frac{\rho}{c}f\left(\frac{\partial(\frac{1}{2}W^2+i)}{\partial t}\right)_\eta=0,$$

and, in the latter case, to

$$\left(\frac{\partial(\rho f)}{\partial t}\right)_{\xi,\eta}\mp\frac{\rho}{c}f\left(\frac{\partial W}{\partial t}\right)_{\xi,\eta}=0.$$

The case $ds/dt\neq0$ is also considered.

R. Finn (Stanford, Calif.)

6895:

Grigorian, S. S. Cauchy's problem and the problem of a piston for one-dimensional, non-steady motions of gas (automodel motion). J. Appl. Math. Mech. 22 (1958), 244–255 (179–187 Prikl. Mat. Meh.).

An interesting, but condensed, generalized treatment is given of the unsteady automodel motion in one space dimension of a polytropic gas under perfect, but non-homentropic, flow conditions. Automodel motions are ones in which the initial and boundary conditions provide only two-dimensional constants; they were introduced by L. I. Sedov [Prikl. Mat. Meh. 9 (1945), 293–311; MR 8, 106]. For these motions the governing equations can be

reduced to a system of ordinary differential equations in which the independent variable is closely related to the Lagrangian particle coordinate. It is shown that, of the explosion and piston problems in one, two and three dimensions considered by earlier authors, many result when specific values are given to the two non-dimensional parameters which are involved.

The topological properties of the solution curves of the system are studied in detail and it is shown that, for a range of initial conditions, either there exists a continuous solution or else there is a discontinuous one satisfying the shock jump conditions. However it is also shown that, for certain ranges of initial conditions, the solution of the initial value problem cannot be continued indefinitely even if moving shocks are allowed. Analysis reveals that the breakdown occurs in the explosion problem when there is a (physically unrealizible) infinite concentration of energy at the origin. Correspondingly for the piston problem, the breakdown occurs when an infinite (unrealizible) pressure arises on the piston.

A. F. Pillow (Toronto)

6896:

Korobeinikov, V. P.; and Riasanov, E. V. Construction of exact discontinuous solutions of the equations of one-dimensional gas dynamics and their applications. J. Appl. Math. Mech. **22** (1958), 362–367 (265–268 Prikl. Mat. Meh.).

A family of exact solutions of the unsteady non-homentropic gas equations in one space variable has been found by L. I. Sedov. [Dokl. Akad. Nauk. SSSR **90** (1953), 735; MR **15**, 175]. The authors use this family to construct a family of discontinuous solutions satisfying the shock jump conditions at a front which advances into a stagnant region of variable density. Some of the simpler cases are discussed briefly and a treatment of blast and piston problems in one, two and three dimensions is sketched.

A. F. Pillow (Toronto)

6897:

Sretenskiĭ, L. N. On the theory of gas jets. Soviet Physics. Dokl. **119** (3) (1958), 252–254 (1113–1114 Dokl. Akad. Nauk SSSR).

It is pointed out that the hodograph equation for flow of a polytropic gas has solutions of the form

$$\psi=e^{\pm n\theta}\exp(\tfrac{1}{2}in\log\tau)F(in,\tau)=e^{\pm n\theta}(T_n'(\tau)+iT_n''(\tau)),$$

where F is a hypergeometric function. An integral representation is then stated for an arbitrary function of τ, over a subsonic range, in terms of a kernel $\{T_n'(\tau_2)T_n''(\tau)-T_n''(\tau_2)T_n'(\tau)\}\{1-(2\beta+1)\tau\}^{\frac{1}{4}}(1-\tau)^{-(1+2\beta)/4}$. Finally a formula is given for efflux of the gas from a nozzle whose walls each consist of two straight parts. There are no proofs, and the statement of the integral representation formula is not quite clear; for example, its range of validity is not stated, through common sense puts it as $(0<\tau\leqq\tau_2)$. Comparison with the formula of Germain [J. Rational Mech. Anal. **4** (1955), 925–941; MR **17**, 846] suggests analogy with the Bessel function formulae of Weber and Hankel, respectively.

T. M. Cherry (Melbourne)

6898:

★Spreiter, John R.; and Alksne, Alberta Y. Aerodynamics of wings and bodies at Mach number one. Proceedings of the Third U. S. National Congress of Applied Mechanics, Brown University, Providence, R.I., June 11–14, 1958, pp. 827–835. American Society of Mechanical Engineers, New York, 1958. xxvii+864 pp. $20.00.

Les auteurs proposent une méthode simple pour évaluer la répartition de pressions sur des obstacles variés (profils, corps de révolution, configuration proprement tridimensionelle) en régime sonique. L'équation du potentiel des vitesses de la théorie des petites perturbations est d'abord linéarisée, en remplaçant, provisoirement, par une constante un facteur variable; puis on applique à la solution de ce problème une méthode de "variation de la constante". Ce procédé peut être selon les cas appliqués de diverses manières; après le choix de la constante d'intégration, on obtient des expressions explicites particulièrement simples pour les répartitions de pressions. De nombreux exemples sont traités et comparés soit à des solutions théoriques exactes, soit à des résultats expérimentaux. Les concordances constatées sont remarquablement satisfaisantes.

P. Germain (Paris)

6899:

Sakurai, Takeo. Flow past a bent flat plate with an unsymmetric dead air at Mach number 1. J. Phys. Soc. Japan **13** (1958), 1055–1060.

The special case treated is the one for which the on-coming stream meets the leading edge of the plate tangentially and for which a dead-air wake is formed by streamlines separating tangentially from the plate at the sharp corner on one side and at the trailing edge on the other. The method of solution is the WKB method used by Imai [J. Math. Phys. **28** (1949), 173–182; MR **11**, 222].

W. R. Sears (Ithaca, N.Y.)

6900:

Sauer, Robert. Überschallströmung um Rumpf-Flügel-Anordnungen. Z. Angew. Math. Phys. **9b** (1958), 601–605.

The author considers a generalization of von Kármán's theory of linearized supersonic steady flow about slender bodies of revolution by superposing higher order singularities in addition to the usual sources and sinks. In this way solutions can be obtained for indented bodies as well as for bodies of revolution. A particular application is to the problem of wing-body interference, in which the reaction of the wing is reduced to a small deformation of the body.

D. Gilbarg (Stanford, Calif.)

6901:

Mahony, J. J. The internal flow problem in axi-symmetric supersonic flow. Philos. Trans. Roy. Soc. London. Ser. A. **251** (1958), 1–21.

The author discusses improvements of the linearized theory for supersonic flow inside axisymmetric ducts and jets. In particular, he considers the case of an expansive discontinuity in the slope of the wall at the entrance of the duct. The expansion fan produced by the discontinuity is approximately the same as a two-dimensional Prandtl-Meyer fan at the corner, but the problem is to continue the solution to the axis and study its reflection there. In the linearized theory the expansion fan is collapsed into a discontinuity in the velocity on the leading characteristic; the discontinuity becomes infinite at the axis and the disturbance is reflected as a logarithmic singularity. This is not the true behavior. A valid approximate solution is found in which the dependence of the flow variables on characteristic variables is the same as in the linearized theory, but the relation of these variables to the space variables is more accurate than in the linearized theory and depends on the solution. In this way crucial non-linear effects are included. The author claims that, in contrast to the external flow problem, it is also essential to modify the boundary conditions in solving

the linearized problem; but, clearly, there is no essential difference in the techniques for the two problems near the wall, and the apparent difference is due to the particular choice of characteristic variables.

In the improved theory, the velocity remains finite and continuous on the axis, the disturbance growing initially like $z^{\frac{1}{2}}$ (where z measures distance along the axis), in contrast to the $z^{-\frac{1}{2}}$ behavior in the linearized theory. It is also found that a shock wave will be formed in the reflected wave. *G. B. Whitham* (New York, N.Y.)

6902:

Vaglio-Laurin, Roberto; and Ferri, Antonio. Theoretical investigation of the flow field about blunt-nosed bodies in supersonic flight. J. Aero/Space Sci. **25** (1958), 761–770.

Numerical solutions are described for axisymmetric flow. An analytic detached shock wave is prescribed, and the corresponding body determined; the direct problem of a prescribed body can then be handled either by systematic modification of the shock or (for slight changes) a linearized perturbation.

In some cases the sonic line can be found completely by approaching it from the supersonic side using the numerical method of characteristics. Then the subsonic region is calculated by marching with a finite difference scheme from the shock to the body. This initial-value problem is badly posed in the sense of Hadamard, but a theorem of Pucci is cited to suggest that the consequent numerical instability can be controlled.

Numerical results are given for one example of a paraboloidal shock wave; the Mach number and adiabatic exponent, though not prescribed, would appear to be ∞ and 7/5. A perturbation of the axisymmetric flow is discussed for treating infinitesimal angle of attack.

M. D. Van Dyke (Paris)

6903:

Utkin, A. I. An investigation of supersonic flows by means of the Volterra integral. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 369–372. (Russian)

Linearized supersonic flow is considered past an axisymmetric shape of nearly constant radius. The integral equation obtained by Volterra's method involves complete elliptic integrals; solution by operational methods or by iteration is proposed. Thus approximate solutions are found for the wave drag of a ring wing of triangular profile, and the shape of the wake behind a cylindrical base. *M. D. Van Dyke* (Paris)

6904:

Kackova, O. N.; and Šmyglevskiĭ Yu. D. Axi-symmetric supersonic flow of a freely expanding gas with a plane transition surface (tables). Vyčisl. Mat. **2** (1957), 45–89. (Russian)

A stream of perfect gas flowing at sonic speed emerges from a circular tube and expands freely. The subsequent flow field has been computed numerically for an adiabatic exponent of 1.14, 1.33, 1.40, and 1.66667, and is tabulated in detail. The solution is started with a series expansion near the sonic disk, and then continued by the numerical method of characteristics. *M. D. Van Dyke* (Paris)

6905:

Carafoli, Elie; et Horovitz, Beatrice. Les problèmes mixtes des ailes triangulaires, pourvues d'une plaque normale, en courant supersonique (ailes cruciformes). Acad. R. P. Romîne. Stud. Cerc. Mec. Apl. **9** (1958), 819–832. (Romanian. Russian and French summaries)

6906:

Paloş, M. M. Calcul analytique d'une classe d'effuseurs supersoniques plans, à pente réduite à proximité du col sonique. Acad. R. P. Romîne. Stud. Cerc. Mec. Apl. **9** (1958), 545–558. (Romanian. Russian and French summaries)

6907:

Carafoli, Elie; et Năstase, Adriana. L'étude des ailes triangulaires minces, à symétrie forcée, en courant supersonique. Acad. R. P. Romîne. Stud. Cerc. Mec. Apl. **9** (1958), 833–853. (Romanian. Russian and French summaries)

6908:

Šefter, G. M. An asymptotic solution of equations for one-dimensional unsteady motion of ideal gas with cylindrical symmetry. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 572–575. (Russian)

Cylindrical blast waves are treated by the method previously used by Yu. L. Yakimov [Prikl. Mat. Meh. **19** (1955), 681–692; MR **17**, 1250] for spherical blast.

J. H. Giese (Aberdeen, Md.)

6909:

Kanwal, R. P. On curved shock waves in three-dimensional gas flows. Quart. Appl. Math. **16** (1958), 361–372.

Les méthodes de la géométrie différentielle sont appliquées à l'analyse de l'écoulement d'un fluide parfait compressible au voisinage d'une onde de choc courbe. L'auteur utilise un système de coordonnées curvilignes bien adapté à cette étude. Il détermine les dérivées des composantes de la vitesse en aval du choc, de la pression, de la masse spécifique de l'entropie, les composantes du tourbillon et caractérise les surfaces de choc derrière lesquelles l'écoulement est irrotationnel si l'écoulement amont est uniforme. *P. Germain* (Paris)

6910:

Pack, D. C. The reflection and transmission of shock waves. I. The reflection of a detonation wave at a boundary. Phil. Mag. (8) **2** (1957), 182–188.

The problem considered is that of transmission and reflection of plane shock waves incident normally on a plane interface between two media. The author states that he assumes the media are barotropic, with the pressure a monotonic increasing function of density. The reviewer notes that a term such as 'piecewise barotropic' would be more accurate, as the author applies his theory, correctly, to media such as air which are not barotropic. A general criterion is obtained for the nature, expansion or shock, of the reflected wave in terms of the shock impedances (product of shock velocity and density ahead of the shock). A further treatment of a particular problem is necessary because the impedance of the transmitted shock is unknown. The case of strong detonation waves is treated and a criterion obtained in terms of known density ratios. Several typical explosion problems are considered.

J. J. Mahony (Sydney)

6911:

Pack, D. C. The reflection and transmission of shock waves. II. The effect of shock waves on an elastic target of finite thickness. Phil. Mag. (8) **2** (1957), 189–195.

A theory is presented for calculating the motion of an elastic target of finite thickness which is unconstrained when it is struck normally by an extremely strong shock wave. It is also assumed that the medium in which the

incident shock is propagating is such that the theory of part I (reviewed above) may be applied. The solution consists of repeated solutions of the one-dimensional wave equation and the application of results of part I. It is shown that each face of the target is subject to a series of impulsive increases of velocity of gradually diminishing magnitude. *J. J. Mahony* (Sydney)

6912:

Theodorides, Phrixos. Parallel effects of bulk viscosity and time lag in kinetics of non-monatomic fluids. Z. Angew. Math. Phys. **9b** (1958), 668–686.

This is a general discussion of the influence of bulk viscosity and relaxation in polyatomic gases, with particular reference to steady one-dimensional shock waves. The equations are presented with quadratic terms representing non-linear viscosity. There are references to calculations without the details.

D. Gilbarg (Stanford, Calif.)

6913:

Lidov, M. L. On limit solutions in the vicinity of a singular point. Dokl. Akad. Nauk SSSR **120** (1958), 1124–1127. (Russian)

The author considers a strong explosion at a point in a medium with variable density and describes the resulting gas motion by relating it linearly to a corresponding motion in a medium with constant density in such a way that the appropriate boundary conditions on the shock wave and at the center of the explosion are simultaneously satisfied. *J. F. Heyda* (Cincinnati, Ohio)

6914:

Ray, G. Deb. An exact analytic solution of equations for an explosion with spherical symmetry. Proc. Nat. Inst. Sci. India. Part A. 23 (1957), 420–429.

The author investigates similarity solutions of the spherical and cylindrical blast wave problems, the wave being headed in each case by a shock of finite strength. As usual, the similarity requirement imposes strict conditions upon the variation of density and pressure in the undisturbed region. In particular, the pressure must fall off as the inverse cube and inverse square of the distance in the spherical and cylindrical cases, respectively, and this requires, in turn, that the shock strength remain constant.

The forces (e.g., gravitation) which determine the equilibrium configuration are neglected when considering the shock propagation. This seems reasonable when the shock is not too weak. *J. Hazlehurst* (Manchester)

6915:

Kochina, N. N.; and Mel'nikova, N. S. Strong point-blasts in a compressible medium. J. Appl. Math. Mech. 22 (1958), 1–19 (3–15 Prikl. Mat. Meh.).

Consider non-steady ν-dimensional ($\nu=1, 2, 3$) radially symmetrical flow of a medium with internal energy $\varepsilon(p, \rho)=(p/\rho_0)\phi(\rho/\rho_0)+\text{const.}$ and with $p=\Psi(S)\chi(\rho/\rho_0)$, where the relation between ϕ, χ, and ρ/ρ_0 is determined by the first law of thermodynamics. For motion behind a shock advancing into a region of zero pressure there are self-similar solutions of the type $v=V(\lambda)r/t$, $\rho=\rho_0 R(\lambda)$, $p=P(\lambda)\rho_0 r^2/t^2$, where $\lambda=r/r_2$ and r_2 is the radius of the shock, given by $r_2 t^{-\delta}=\text{const.}$, with $\delta=2/(2+\nu)$. V, R, and P satisfy a system of ordinary differential equations which can be reduced by means of two integrals, the energy and adiabatic equations, to a single first order equation (E) for R as a function of V. For the three choices (i) $\phi(R)=(R-1)/2R^2$; (ii) $\phi(R)=(AR^2+B)/R(C+R^2)$, A, B, C constants; (iii) $\phi(R)=1$, the author makes an analysis of singular points of (E) and determines behavior in the large of the integrals of (E) to show existence of self-symmetrical flows which extend all the way from the shock to $r=0$, or which end at a spherical cavity. Results of calculations for various shock strengths and other choices of parameters are given. The author also develops a criterion for determining the range of parameters for which these point blast results are valid.

J. H. Giese (Aberdeen, Md.)

6916:

Savage, James C. Reflection from a fluid of higher sound velocity. J. Acoust. Soc. Amer. **30** (1958), 974–978.

An exact solution, in the form of an elliptic integral, is obtained for the pressure response to a spherically symmetric acoustic pulse originating at a source within a semi-infinite fluid bounded below by a fluid of higher sound velocity. It is somewhat remarkable that the author does not refer to Sommerfeld's work on the dipole antenna over the earth [Ann. Physik (4) **28** (1909), 665–736] which leads to the same mathematical problem. The author's contribution is to use a method developed by L. Cagniard [Ondes seismiques progressives, Gauthier-Villars, Paris, 1939] to invert the Laplace transform of the solution. This method enables him to analyze the refracted arrival and the total pressure response.

I. Stakgold (Washington, D.C.)

6917:

Davis, L.; Lüst, R.; and Schlüter, A. The structure of hydromagnetic shock waves. I. Non linear hydromagnetic waves in a cold plasma. Z. Naturf. **13a** (1958), 916–936.

Les auteurs considèrent les équations macroscopiques de la magnétoaérodynamique et recherchent les solutions qui dépendent seulement de l'abscisse. Les mouvements des ions et des électrons sont considérés individuellement, la divergence du champ électrique étant proportionnelle à la différence des nombres spécifiques ionique et électronique. Avec ces hypothèses, les équations du problème peuvent être intégrées au moyen des fonctions elliptiques. On trouve ainsi des trains d'ondes (solutions périodiques) et des ondes solitaires (solutions non périodiques); toutes les ondes peuvent être incluses dans une famille à deux paramètres.

Ces ondes sont en rapport avec les ondes de choc dans les plasmas de faible densité. Deux relations, analogues à celles de Rankine et Hugoniot, sont obtenues; pour l'une d'elles, les régions situées de part et d'autre du choc contiennent des trains d'ondes différents; l'autre est une relation classique de choc hydromagnétique normal.

H. Cabannes (Marseille)

6918:

Agostinelli, Cataldo. Sui vortici sferici in magneto idrodinamica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **24** (1958), 35–42.

Les équations de la magnétohydrodynamique pour un fluide incompressible, non visqueux, conducteur électrique parfait, sont étudiées dans le cas axisymétrique. Une solution est obtenue pour les fonctions de courant représentant les parties poloïdales des champs. Il est possible de particulariser cette solution de façon à obtenir la représentation d'un tourbillon sphérique de Hill. On étudie alors la distribution de la pression et du champ magnétique à l'extérieur du tourbillon.

On démontre que "dans un fluide indéfini homogène incompressible, électriquement conducteur et de con-

ductivité infinie, soumis à un champ magnétique uniforme, il peut se former un tourbillon sphérique magnétohydrodynamique, dont le centre est animé d'un mouvement rectiligne uniforme dans la direction du champ magnétique uniforme extérieur". *J. Naze* (Marseille)

6919:

Agostinelli, Cataldo. Figure di equilibrio ellissoidali per una massa fluida elettricamente conduttrice uniformemente rotante, con campi magnetici variabili col tempo. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **23** (1957), 409–414.

On recherche la possibilité d'un mouvement de rotation uniforme autour d'un de ses axes, O_z, pour une masse ellipsoïdale de fluide incompréssible, non visqueux, électriquement conducteur, soumis à sa propre gravitation et à un champ magnétique. A l'intérieur de la masse fluide les composantes du champ magnétique par rapport à des axes fixes O_{xyz} sont supposées dépendre linéairement des variables d'espace x, y et périodiquement du temps, avec une période égale à celle de la rotation. Les équations du mouvement sont intégrées, et les différents cas discutés; l'ellipsoïde n'est en général pas de révolution. On démontre enfin que la limite supérieure de la vitesse angulaire de rotation est plus grande en présence d'un champ magnétique à l'intérieur de la masse fluide, que dans le cas classique. *J. Naze* (Marseille)

6920:

Gershuni, G. Z.; and Zhukhovitskii, E. M. Stationary convective flow of an electrically conducting liquid between parallel plates in a magnetic field. Soviet Physics. JETP 34(7) (1958), 461–464 (670–674 of Russian original).

An explicit solution is given to the equations of convective flow of a conducting fluid between two parallel vertical plates held at different temperatures in the presence of a uniform, constant magnetic field applied perpendicularly to the plates. The appearance of a thin boundary layer with large velocity gradients near the plates occurs with high magnetic fields. The magnetic field reduces the magnitude of the velocity itself. The assumption of a simple Ohm's law in the form $\mathbf{j}=\sigma(\mathbf{E}+c^{-1}\mathbf{v}\times\mathbf{B})$ may be unrealistic.

A. A. Blank (New York, N.Y.)

6921:

Gershuni, G. Z.; and Zhukhovitskii, E. M. Stability of the stationary convective flow of an electrically conducting liquid between parallel vertical plates in a magnetic field. Soviet Physics. JETP 34(7) (1958), 465–470 (675–683 of Russian orignal).

The stability of the flows of the preceding paper is investigated by the conventional linearized perturbation method. It is found that, in addition to reducing the overall velocity, the magnetic field reduces the rate of growth of the unstable modes.

A. A. Blank (New York, N.Y.)

6922:

Newcomb, William A. Motion of magnetic lines of force. Ann. Physics 3 (1958), 347–385.

A simple and clean-cut treatment of the concept of velocity of a magnetic line of force. To ascribe a velocity v to magnetic lines of force makes sense if and only if the flux through an arbitrary cycle moving with the velocity $\mathbf{v}$ is constant throughout the motion. A necessary and sufficient condition for the constancy of flux is $\nabla\times(\mathbf{E}+\mathbf{v}\times\mathbf{H})=0$. If $\mathbf{v}$ is the drift velocity $\mathbf{E}\times\mathbf{H}/H^2$ of an ionized particle, then this condition is equivalent to the requirement $\nabla\times[\mathbf{H}(\mathbf{E}\cdot\mathbf{H})/H^2]=0$ which is automatically satisfied if E is perpendicular to H. When the particle drift velocity is not flux-preserving there may yet exist special flux-preserving cycles, and such cases also are considered. *A. A. Blank* (New York, N.Y.)

6923:

Le Claire, A. D. Random walks and drift in chemical diffusion. Phil. Mag. (8) 3 (1958), 921–939.

OPTICS, ELECTROMAGNETIC THEORY, CIRCUITS

See also 6824, 6916, 6917, 6922.

6924:

Havliček, F. I. Zur praktischen Anwendung der Wellentheorie bei optischen Rechnungen. Bull. Internat. Acad. Yougoslave. Cl. Sci. Math. Phys. Tech. **5** (1955), 5–7.

If the spherical aberration of an axially symmetrical optical system is $\Delta s^1=\sum_{n=1}^{\infty} a_n u^n$, where $u=\tan\theta$ and θ is the angle of inclination of an incident ray, then J. Picht [Optische Abbildung, Vieweg, Braunschweig, 1931, p. 164; Z. Instrumentenkunde **51** (1931), 77–81] has shown that the related wave surface in image space is given by

$$\xi=-2b_1+c_1\cos\theta+\sum_{n=1}^{\infty} b_n u^n,$$
$$\eta=2\cos\phi\{c_2\sin\theta-\sum_{n=1}^{\infty} nb_n u^n/(2n-1)\},$$
$$\zeta=2\sin\phi\{c_3\sin\theta-\sum_{n=1}^{\infty} nb_n u^n/(2n-1)\},$$

where c_1, c_2, c_3 are arbitrary constants and $2nb_n=(2n-1)(a_{n-1}-b_{n-1})$ is a recursion formula for the coefficients (reviewer's notation). The present paper contains a new proof of this result and an application showing that a particular Kirchhoff diffraction integral can be evaluated in terms of a well-known transcendental function. There are too many misprints in the paper to list here, but they are fortunately evident from the above reference.

G. L. Walker (Southbridge, Mass.)

6925:

Siegel, K. M. Far field scattering from bodies of revolution. Appl. Sci. Res. B 7 (1958), 293–328.

This paper is a summary of the extensive work of its author and others on the calculation of approximate electromagnetic scattering cross sections of perfectly conducting bodies of revolutions. Most of the results are for incidence along the axis of symmetry. Use is made of the Rayleigh, Kirchhoff, Fock, and Franz approximations in the appropriate wavelength regimes, and combinations of these methods are applied in some special cases. Numerous examples are given, and the results are compared with exact calculations or experimental results when possible. *E. T. Kornhauser* (Providence, R.I.)

6926:

Seckler, Bernard D.; and Keller, Joseph B. Geometrical theory of diffraction in inhomogeneous media. J. Acoust. Soc. Amer. 31 (1959), 192–205.

This paper starts with a discussion of geometrical-optics approximations in general. In optical approximations the phase of the wave function depends on the optical length $\int n ds$ (n=refractive index) of the relevant trajectory; the amplitude A follows from the conservation law according to which the energy flux $\iint nA^2 d\sigma$ is identical for all cross-sections of a tube limited by ray

trajectories. The field of applications can be extended by also introducing, along with the conventional trajectories, the "diffracted rays" and "imaginary rays" defined in this article, as explained below.

Let us consider an ordinary ray which is tangent to a material boundary. A part of the energy arriving near the point of tangency proceeds along the produced ordinary ray, another part follows the boundary as a "surface ray". At each further point of this ray another part of the energy is lost, this part being propagated away from the surface along the tangent at the point under consideration. Such rays tangential to the boundary surface are termed "diffracted rays". An application of the above conservation law alone would involve an infinite amplitude along a surface ray, since the latter constitutes a caustic for the associated diffracted rays. However, plausible assumptions are introduced in order to arrive at extensions of the conventional geometrical optics approximations which are also finite along the surface ray and the diffracted rays. For instance, the decrease of the amplitude along a surface ray is supposed to be exponential, apart from the effects due to a divergence of these rays over the boundary surface. Moreover, the relevant parameters are assumed to depend only on the sum of the local curvatures of the tangential ray and that of the normal section of the surface containing this tangent. The analytical dependence on this total curvature is chosen in accordance with asymptotic forms (for large wave numbers) of the known rigorous solutions for special diffraction problems (convex cylinder, sphere, etc.).

The "imaginary" or "complex" rays play a role in shadow regions which are inaccessible to ordinary trajectories. These rays are defined as complex-valued solutions of the ray equation, and consist of a sequence of points with complex coordinates, the real parts of which are in the same domain as the actual coordinates of the shadow region in question. An ordinary ray reaching the shadow limit from the lit side is split into a real ray continuing along the produced trajectory, and a complex ray penetrating into the shadow; on the other hand, a complex ray arriving from the shadow region is split at the shadow limit into another complex ray returning towards the shadow, and an ordinary ray proceeding in the lit region. These splits are associated with changes of the amplitude and phase that are chosen in accordance with the asymptotic forms of the rigorous solution of the wave equation.

The extended geometric-optical theory including the effects of "diffracted rays" and "complex rays" is applied to stratified media. The cases of an unbounded medium and of a plane and a cylindrically stratified medium (limited by a plane or cylinder) are discussed. The boundary condition $\partial u/\partial x=ikZu$ (impedance condition, or Leontovich boundary condition) is then assumed to hold along the boundary. The field is assumed generated by a plane incident wave, or by a line or point source. In all these cases the final results are proved to agree with those obtained from the asymptotic forms of the corresponding rigorous solutions of the wave equation; these latter forms are represented in a comparison article by the same authors [reviewed below]. The correspondence between these asymptotic forms and the results of the extended geometrical optics justifies the assumptions underlying the latter.

H. Bremmer (Eindhoven)

6927:

Seckler, Bernard D.; and Keller, Joseph B. Asymptotic theory of diffraction in inhomogeneous media. J. Acoust. Soc. Amer. 31 (1959), 206–216.

This article discusses many results of the conventional theories for wave propagation through media which are plane or cylindrically stratified in an x or r direction. Explicit formulas are given for the asymptotic expressions for the modes obtained by separation of coordinates, and for final field strengths depending on an integration over these modes. All asymptotic expressions refer to large wave numbers k and are essentially W.K.B. approximations. The medium extends either to infinity in all directions, or it is bounded at one side by a plane $x=b$ or a cylinder $r=(x^2+y^2)^{\frac{1}{2}}=b$. The boundary condition assumed at the latter is of the form $\partial u/\partial x=ikZu$ (so-called impedance condition, or Leontovich boundary condition). The field is assumed as generated by a plane incident wave, or by a line or point source.

The conditions of symmetry involve mode solutions of the types $\exp(ikn_0ay)f(x, a)$, $H_0^{(1)(2)}(kar)f(x, a)$, and $f(r, m)\exp(im\theta)$, respectively. The factor $f(x, a)$ satisfies the differential equation:

$$(1) \qquad \frac{\partial^2 f}{\partial x^2}+k^2(n^2(x)-a^2)f=0,$$

which depends on the complete profile $n(x)$ of the refractive index. The analytical form of the asymptotic expressions for $f(x, a)$ is connected with the sign of $n^2(x)-a^2$; therefore, the final representations differ in various parts of space and for various ranges of the separation parameter a.

In the region lit by the source a saddlepoint approximation (principle of stationary phase) can be applied to the integral which represents the total field strength in terms of modes; this results in geometric-optical expressions of the conventional form. In shadow regions the same integral can be reduced to a convenient form by transforming its path of integration into a proper complex contour (method of "Watson transformations"). The enclosed poles involve a residue series, whereas the combination of all additional contributions (including the one resulting from an occasional cross cut of $f(x, a)$ as a function of a) are proved to disappear asymptotically. The poles in question are the zeros of the Wronskian of $f_1(x, a)$ and $f_2(x, a)$; f_1 and f_2 here represent solutions of (1) that satisfy the radiation condition at infinity and the impedance condition at the finite boundary, respectively.

The results arrived at agree with those derived in a companion article by the same authors [reviewed above]. The tentative principles introduced in this latter article are thus verified for very general circumstances.

H. Bremmer (Eindhoven)

6928:

Isakovič, M. A. Scattering of sound waves on small inhomogenities in a waveguide. Akust. Ž. 3 (1957), 37–45. (Russian)

The method of small excitations is applied to the determination of the field of a singly scattered wave in a planar waveguide with small inhomogeneities. The problem is solved for the waveguide filled with material whose index of refraction fluctuates from point to point and also for a waveguide with rough walls and homogeneous medium. The scattered field is found in the form of superposition of normal waves for the waveguide free of inhomogeneities.

C. H. Papas (Pasadena, Calif.)

6929:

Partearroyo, R. Alcain. The generalized theorem of Pythagoras and its realization in an electric network. Calc. Automat. y Cibernet. 7 (1958), no. 19, 40–46. (Spanish. French summary)

CLASSICAL THERMODYNAMICS, HEAT TRANSFER

See 6826, 6893.

QUANTUM MECHANICS

See also 6710, 6821, 6822.

6930:

Green, H. S. Observation in quantum mechanics. Nuovo Cimento (10) 9 (1958), 880–889.

In what sense does a quantum mechanical system "jump" into an eigenstate of an observable when an observation is made? This question troubled the inventors of quantum mechanics and has recently been the subject of extensive discussions. [See, for example, H. Everett, Rev. Mod. Phys. 29 (1957), 454–462; MR 20 # 679]. The present author here takes the view that the act of measurement is indeed irreversible, because the macroscopic apparatus necessary for the measurement undergoes a thermodynamically irreversible transition during the measurement. He supports it by constructing a model experiment in which the spin of a particle is coupled to two detectors each consisting of two sets of harmonic oscillators. The sets of oscillators are at two different temperatures and in the act of observation come to new equilibrium temperatures.

A. S. Wightman (Princeton, N.J.)

6931:

Nakanishi, Noboru. A theory of clothed unstable particles. I, II. Progr. Theoret. Phys. 19 (1958), 607–621; 20 (1958), 822–834.

The author's summary of the first paper reads: "The clothed states of unstable particles are investigated on the basis of quantum field theory, and thereby a new mathematical notion, "complex distribution", is introduced. Then, an exact eigenstate of the total Hamiltonian, with a complex eigenvalue whose real part represents the mass of the unstable particle and whose imaginary part is interpreted as half the reciprocal of its life time, can be constructed by means of the complex distributions. But this state is not observable. The physical state of the unstable particle is defined as an approximate state of the exact eigenstate which exhibits a physically reasonable behaviour." The "complex distribution" mentioned here appears to the reviewer to be only a somewhat elaborate way of writing the conventional residue calculus and the "exact eigenstate" of the Hamiltonian with a complex eigenvalue obtained this way has, as mentioned in the paper, very little significance. The summary of the second paper reads: "The physical state of an unstable particle previously proposed is derived as the state produced in a finite time interval on the basis of the scattering theory of Gell-Mann and Goldberger, and also the exact state is obtained from the physical state by employing an appropriate limiting process. An S-matrix-theoretical calculation method is proposed for processes including unstable particles." The general philosophy of the author seems to be that his "physical state" of the unstable particle needs a justification, while the exact eigenstate with complex eigenvalue is a good concept.

G. Källén (Lund)

6932:

Polkinghorne, J. C. Causal amplitudes and the Yang-Feldman formalism. Proc. Cambridge Philos. Soc. 53 (1957), 843–847.

The author defines a free field $\phi_F(x)$ with the aid of the equation

$$\phi(x)=\phi_F(x)+\int\Delta_F(x-x')j(x')dx',$$

where the "current" $j(x)$ is defined as $(\Box-\mu^2)\phi(x)=j(x)$ and where Δ_F is the singular function of Feynman. The Fourier components of the field ϕ_F agree with the Fourier components of the incoming field for positive frequencies and with those of the outgoing field for negative frequencies. From this it follows at once that the S-matrix of the theory can be expressed in terms of ϕ_F. In this way, the author rederives some well-known results for the S-matrix.

G. Källén (Lund)

6933:

Rideau, Guy. Sur l'introduction des opérateurs habillés en théorie quantique des champs. C. R. Acad. Sci. Paris 247 (1958), 1098–1101.

"Avec certaines hypothèses sur les spectres des hamiltoniens, sont construits les vecteurs représentant les particules physiques. On en déduit une définition des éléments de matrice de la matrice S ne faisant plus intervenir que les opérateurs de création et d'annihilation des particules physique." (Résumé de l'auteur)

G. Källén (Lund)

6934:

Vrkljan, V. S. Über die Anwendung der Methode des Darwinschen Wellenpaketes in der de Brogliesschen Theorie der Partikeln mit dem Spin 1 vom Typus des Mesons. Bull. Internat. Acad. Yougoslave. Cl. Sci. Math. Phys. Tech. 5 (1955), 33–34.

Es handelt sich um Partikel, welche durch die Wellengleichung

$$(*)\qquad \mathfrak{H}\psi_{kl}+\frac{h}{i}\frac{a_4+b_4}{2}\frac{\partial\psi_{kl}}{\partial t}=0$$

charakterisiert werden. $\mathfrak{H}$ ist der Hamiltonsche Operator, wie er in die neuere Theorie durch L. de Broglie eingeführt worden ist [cf. L. de Broglie, Théorie générale des particules à spin, Gauthier-Villars, Paris, 1943; p. 101]. (*) genügt nicht, um das magnetische Moment der de Brogliesschen Partikel zu bestimmen und muß durch weitere Gleichungen ergänzt werden. Der Berechnung des gesuchten magnetischen Momentes geht die Berechnung der Komponenten der statistischen Dichte des elektrischen Stromes voraus. Die in die Rechnung eingehenden Diracschen Matrizen entsprechen de Broglie's Methode der Fusion.

M. Pinl (Cologne)

6935:

Vrkljan, V. S. Führen die de Brogliesschen Teilchen mit dem Spin 1 vom Typus des Mesons die Schrödingerschen Oszillationen aus? Bull. Internat. Acad. Yougoslave. Cl. Sci. Math. Phys. Tech. 5 (1955), 35–38.

Entgegen einem Theorem von Ehrenfest führt nach E. Schrödinger der statistische Schwerpunkt des Elek-

trons und des Positrons entsprechend der Diracschen Theorie eine Serie von Oszillationen aus. Auch für de Brogliesche Partikel vom Spin 1 vom Typus des Mesons gilt Ehrenfest's Theorem nicht. Verfasser errechnet unter Verwendung der de Broglieschen Fusionsmethode einen analogen Oszillationseffekt für diese Mesonen.
M. Pinl (Cologne)

6936:
Vrkljan, V. S. Über die Spin-Operatoren in der neueren de Broglieschen Theorie der Partikeln mit dem Spin-Maximum 1. Bull. Internat. Acad. Yougoslave. Cl. Sci. Math. Phys. Tech. **5** (1955), 49–52.

In einer früheren Theorie gelangte L. de Broglie zu Spinoperatoren in der Partikeltheorie vom Spinmaximum 1 durch Bestimmung des Operators des gesamten Impulsmomentes, der mit dem Hamiltonschen Operator vertauschbar ist [cf. L. de Broglie, Une nouvelle théorie de la lumière, la mécanique ondulatoire du photon, tom I, La lumière dans le vide, Hermann, Paris, 1940; sehe S. 185; und Théorie générale des particules à spin, Gauthier-Villars, Paris, 1943; S. 131]. In einer neueren Theorie entwickelte L. de Broglie jedoch zum gleichen Zweck eine andere Methode [cf. L. de Broglie, Mécanique ondulatoire du photon et théorie quantique des champs, Gauthier-Villars, Paris, 1949; MR **10**, 663; pp. 32–33]. Demgegenüber verfolgt Verfasser L. de Broglie's erste Methode auch im Falle von dessen neuerer Theorie und untersucht die auf diesem Wege zu gewinnenden Spinoperatoren. *M. Pinl* (Cologne)

6937:
Vrkljan, V. S. Besteht ein Analogon dem Hertzschen Vektor bei den de Broglieschen Gleichungen des Elektrons und des Positrons? Bull. Internat. Acad. Yougoslave. Cl. Sci. Ma h. Phys. Tech. **5** (1955), 95–96.

L. de Broglie hat aus den Diracschen Gleichungen des Elektrons bzw. Positrons Gleichungen vom Maxwellschen Typus für dieselben Teilchen hergeleitet. Verfasser führt diese Ergebnisse weiter und findet auch das Analogon zum Hertzschen Vektor innerhalb L. de Broglie's Gleichungen vom Maxwellschen Typus. Überdies lassen sich L. de Broglie's erwähnte Gleichungen vom Maxwellschen Typus mit L. de Broglie's "Photonengleichungen" vereinheitlichen. *M. Pinl* (Cologne)

6938:
Bellman, Richard; Kalaba, Robert; and Wing, G. Milton. Invariant imbedding and neutron transport theory. II. Functional equations. J. Math. Mech. **7** (1958), 741–756.

This paper illustrates further the applicability of the authors' concept of "invariant imbedding" [Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 517–520; J. Math. Mech. **7** (1958), 149–162; MR **19**, 506; **20**#2046], to one-dimensional neutron multiplication problems. Both time-dependent and energy-dependent models are treated. A numerical procedure is derived for calculating the critical diameter and energy dependence of idealized "fission products", through the solution of a suitable non-linear differential equation of Riccati type.
G. Birkhoff (Cambridge, Mass.)

6939:
Freeman, A. J.; and Löwdin, P. O. Quantum-mechanical kinetic energy transformation. Phys. Rev. (2) **111** (1958), 1212–1213.

"It is shown that by a transformation of the usual expression for the kinetic energy matrix elements a simpler formula results which yields, for numerical wave functions, higher numerical accuracy for the kinetic energy than has previously been obtained. The derivation of the transformation and the numerical results of several applications are presented." (Author's summary)
A. C. Hurley (Melbourne)

RELATIVITY

6940:
Weber, Joseph; and Wheeler, John A. Reality of the cylindrical gravitational waves of Einstein and Rosen. Rev. Mod. Phys. **29** (1957), 509–515.

The authors concern themselves with the question of the reality of gravitational radiation. Detailed consideration is given to cylindrical solutions of the field equations. It is confirmed that the pseudo-energy density is zero in a cylindrical field, or equal and opposite to all other forms of energy density if these happen to be present. It is further shown that the Riemann tensor of such a field does not vanish in general. The response of test particles to a cylindrical wave is considered, and it is claimed that because the distance between nearby particles changes with time "this change can be used to drive an engine". The authors' conclusion from all this appears to be that reality is to be ascribed to cylindrical gravitational waves. *H. A. Buchdahl* (Princeton, N.J.)

6941:
Eisenhart, Luther P. Spaces for which the Ricci scalar R is equal to zero. Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 695–698.

L'auteur écrit la condition pour qu'un espace de Riemann à 3 dimensions de métrique:

$$ds^2=g_{11}(dx^1)^2+g_{22}(dx^2)^2+g_{33}(dx^3)^2$$

ait une courbure riemanienne scalaire nulle (condition vérifiée par la partie spatielle d'une métrique einsteinienne statique ou symétrique par rapport du temps au sens de J. Weber et J. A. Wheeler [#6940 ci-dessus]. Il obtient diverses solutions sous la forme

$$\sqrt{g_{11}}=x_1^{a_1}x_2^{b_1}x_3^{c_1},\ \sqrt{g_{22}}=x_2^{a_2}x_3^{b_2}x_1^{c_2},\ \sqrt{g_{33}}=x_3^{a_3}x_1^{b_3}x_2^{c_3},$$

ou

$$\sqrt{g_{11}}=x_2^{a_1}+x_3^{b},\ \sqrt{g_{22}}=x_1^{a_2},\ \sqrt{g_{33}}=x_1^{a_3},$$

où les a, b, c sont des constantes.
Y. Fourès-Bruhat (Marseille)

6942:
Costa de Beauregard, Olivier. L'effet gravitationnel de spin. C. R. Acad. Sci. Paris **246** (1958), 237–240.

6943:
Treder, H. Space-time structure of a static spherically symmetric scalar field. Phys. Rev. (2) **112** (1958), 2127.

O. Bergmann and R. Leipnik derived some special solutions of Einstein's field equations for which the energy-momentum tensor is obtained from a particle surrounded by a spherically symmetric static scalar field $V(r)$ [Phys. Rev. (2) **107** (1957), 1157–1161; MR **19**, 1021]. It is shown in the present paper that the solutions obtained by these authors can be valid only if V=const., i.e., for a vanishing scalar field. *H. Rund* (Durban)

ASTRONOMY

6944:

Whitney, Ch.; et Ledoux, P. Note sur le calcul numérique des pulsations stellaires. Acad. Roy. Belg. Bull. Cl. Sci. (5) 43 (1957), 622–627.

The authors have adopted a new approach to the numerical solution of the pulsation problem, whereby the star is considered as being composed of n discrete shells. When the pulsation amplitude is small, the frequencies and amplitudes of the n possible modes of vibration are obtained by solving n linear algebraic equations.

The lumping of the mass becomes inaccurate near the stellar surface, where the conditions change rapidly — a difficulty already familiar in the integration of the partial differential equations of the non-linear problem. In fact, the shell approach is more closely related to the methods based upon the exact hydrodynamical equations than is the conventional procedure of covering all small pulsations by solving a succession of two-point boundary value problems in ordinary differential equations.

On the other hand, the conventional method possesses the advantage that the inevitable surface inaccuracies do not greatly affect the solution in the rest of the star.

J. Hazlehurst (Manchester)

6945:

Danby, J. M. A.; and Camm, G. L. Statistical dynamics and accretion. Monthly Not. Roy. Astr. Soc. **117** (1957), 50–71.

In the first part of the paper, the authors treat the motion of the particles of a cloud in the gravitational field of a point mass with a uniform speed with respect to the cloud. The particles of the cloud are supposed not to interact, either by collision or gravitational forces. At great distances from the attracting point mass, the density is supposed to be constant and the velocity distribution of the particles is supposed to be Maxwellian. The density and velocity distributions at an arbitrary point are derived, and numerical results given for points situated straight in the "windward" and the "leeward" direction as seen from the attracting point mass, the increment of the density due to the attracting mass being much larger on the "leeward" side. The deviation from the Maxwellian velocity distribution is considerable in the neighbourhood of the attracting mass; under usual conditions, the Maxwellian velocity distribution is a good approximation only for distances larger than 200 astronomical units.

In the second part of the paper, the authors discuss the applicability of their model, especially the assumptions of no collisions between the particles, the HI and HII regions being treated separately. They find that the collisions are so rare that the capturing of particles due to collisions is unimportant.

At last the authors present a modification of Eddington's treatment of the problem of accretion of matter in order to allow for the velocity distribution within the cloud. If collisions between the particles are neglected, it is found that Eddington's conclusion that in general the accretion is insignificant is fully confirmed.

In an appendix some computations concerning the collision frequencies in rarified gases at rest in gravitational fields are given.

E. Lyttkens (Uppsala)

GEOPHYSICS

See 6861, 6873.

OPERATIONS RESEARCH AND ECONOMETRICS

See also 6734, 6735, 6736.

6946:

★Bowman, Edward H.; and Fetter, Robert B. (Editors) Analyses of industrial operations. The Irwin Series in Industrial Engineering and Management. Richard D. Irwin, Inc., Homewood, Ill., 1959. viii+485 pp. $7.95.

This volume consists of 26 papers, nearly all reprinted from "Management Science", "Operations Research", and similar journals. Thirteen present applications of mathematical programming to business and industrial problems; the other thirteen illustrate the application of other mathematical procedures to managerial problems. The selection is based on the interest and significance of the application rather than on intrinsic mathematical interest; indeed, there is little mathematics in the narrow sense. The whole is a suggestive sample of the possibilities of applying mathematical methods in business.

R. Dorfman (Cambridge, Mass.)

6947:

Bishop, Albert B.; and Rockwell, Thomas H. A dynamic-programming computational procedure for optimal manpower loading in a large aircraft company. Operations Res. **6** (1958), 835–848.

The authors utilize the functional equation technique of dynamic programming to formulate and resolve computationally some multi-stage scheduling problems. As the authors show, this method is independent of the functional form of the cost relation, can utilize tabular data directly without curve-fitting of any type, can easily be used for sensitivity studies, and is easily programmed for either machine or hand computation.

R. Bellman (Santa Monica, Calif.)

6948:

Vorob'ev, N. N. Equilibrium points in bimatrix games. Teor. Veroyatnost. i Primenen. **3** (1958), 318–331. (Russian. English summary)

Let X, Y be, respectively, mixed strategies for players 1 and 2 of a two-person non-zero-sum game. Suppose (X, Y) has the following property: If player 1 [2] changes his strategy while player 2 [1] does not the payoff to player 1 [2] is decreased. Then (X, Y) is called an equilibrium point of the game. The author describes an algorithm for computing all equilibrium points of the game.

J. Wolfowitz (Ithaca, N.Y.)

BIOLOGY AND SOCIOLOGY

6949:

Macey, Robert I. A quasi-steady-state approximation method for diffusion problems: I. Concentration dependent diffusion coefficients. Bull. Math. Biophys. **21** (1959), 19–32.

An approximation to the diffusion equation was proposed by H. D. Landahl [same Bull. **15** (1953), 49–61;

MR 14, 781] for the case of a constant diffusion coefficient, the approximation representing the process as a sequence of quasi-steady states. With the initial condition $c(x, 0)=0$ for $x\geqq 0$, and the boundary condition $c(0, t)=c_0$ for $t\geqq 0$, in a semi-infinite medium, this provides a vanishing concentration at $x\geqq r(t)$, where $r(t)$ is some increasing function of t. The author extends the method to permit the diffusion coefficient to be a function of c, and compares the results with numerical solutions of the exact equation. Generally speaking, the agreement is quite good except in the neighborhood of $r(t)$.

A. S. Householder (Oak Ridge, Tenn.)

6950:

Williamson, M. H. Some extensions of the use of matrices in population theory. Bull. Math. Biophys. 21 (1959), 13–17.

This paper extends Leslie's vector and matrix treatment of populations. A simple matrix is given for species in which adult mortality and fertility are independent of age, but in which the juvenile mortality rate differs from the adult. The population vector can be changed into a population matrix. This should allow equations using functions of the size of the population to be developed. Genetic variables such as sex or other polymorphisms can be introduced, and the notation allows different systems of selection or non-random mating to be specified.

Author's summary

INFORMATION AND COMMUNICATION THEORY

6951:

Shannon, C. E. Channels with side information at the transmitter. IBM J. Res. Develop. 2 (1958), 289–293.

A stationary memoryless channel (without side information) is characterized by conditional probabilities $P_i(j)$ that symbol j will be received if symbol i is sent, where $i \in A=\{1, 2, \cdots, a\}$ and $j \in B=\{1, 2, \cdots, b\}$. When there is side information one has a channel condition variable t and probabilities $P_{ti}(j)$ that j will be received if the channel is in condition t when i is sent, with $t \in H=\{1, 2, \cdots, h\}$. The value of the channel condition variable is known to the sender before each symbol transmission. The author considers in detail only the case where the successive channel conditions are mutually independent random variables with common probability distribution g_t, $t \in H$. Block codes and channel capacity are defined. It is shown that for these purposes the channel is equivalent to a stationary memoryless channel without side information whose a^h input symbols are the elements of A^H, whose output symbols are the elements of B (as before), and whose input-output probabilities are $r_X(y)=\sum_t g_t P_{tx_t}(y)$, where $X=\{x_1, x_2, \cdots, x_h\} \in A^H$. In an appendix the author proves that if a block code of rate $R>C$ is applied to a stationary memoryless channel (without side information) of capacity C, then

$$P_e \geqq \frac{R-C}{6[R-\ln((R-C)/R)]}$$

is a lower bound for the probability of error per block.

S. P. Lloyd (Murray Hill, N.J.)

6952:

Shannon, Claude E. A note on a partial ordering for communication channels. Information and Control 1 (1958), 390–397.

A stationary discrete memoryless channel is characterized by a (not necessarily square) stochastic matrix $K=\|p_i(j)\|$ of probabilities $p_i(j)$ that output symbol j will be received if input symbol i is sent. The author defines a relation $\supseteq$ by $K_1 \supseteq K_2$ if and only if $K_2=\sum_\alpha g_\alpha R_\alpha K_1 T_\alpha$ for some set of probabilities g_α and (not necessarily square) stochastic matrices R_α, T_α. The following properties are established: (1) $\supseteq$ is a partial ordering of the set of all stochastic matrices (channels); (2) $\supseteq$ is preserved in direct sums and Kronecker (outer) products; (3) for fixed K_1 [and fixed dimensionality for K_2], the set $\{K_2 | K_1 \supseteq K_2\}$ is convex; (4) if $K_1 \supseteq K_2$, then any block code for K_2 yields a block code for K_1 which has probability of error no larger, and (capacity of channel K_1)$\geqq$(capacity of channel K_2); (5) the set of all 2×2 stochastic matrices ordered by $\supseteq$ is a lattice. The author states the following unsolved problems. (a) Does $\supseteq$ give rise to a lattice in higher dimensions? (b) "Can one show that, in some sense, [$\supseteq$] is the most general ordering for which [(4) above] will hold?" (c) What is the most natural generalization to channels with memory?

S. P. Lloyd (Murray Hill, N.J.)

6953:

Béthoux, Paul. Nombre maximum de signaux d'énergie totale fixée parmi lesquels on peut discriminer à ε près en présence d'un bruit blanc Gaussien. C. R. Acad. Sci. Paris 247 (1958), 573–575.

Let a certain channel degrade transmitted signals $\rho(t)$ into received signals

$$V(t)=\rho(t)+U(t),$$

where $U(t)$ is stationary gaussian noise of mean zero and covariance

$$\Gamma(t, \tau)=N(\sin \Omega(t-\tau))/(\Omega(t-\tau)).$$

With signals bandlimited to frequency $\Omega/(2\pi)$ and of average power P, the author asserts that $C\geqq \Omega P/(2\pi N)$ is a lower bound for the capacity of the channel {inconsistent with the Shannon formula $C=(\Omega/(2\pi))\ \ln(1+(P/N))$}. The author describes his methods (involving eigenfunctions of Γ as an integral operator on $L_2(0, T)$), but omits details in the final steps of his proof.

S. P. Lloyd (Murray Hill, N.J.)

6954:

Wiesner, J. B. Communication science in a university environment. IBM J. Res. Develop. 2 (1958), 268–275.

A general discussion of communication sciences. Of possible interest to mathematicians is the following quotation. "Cardinal to progress in all of the communication sciences is the development of the appropriate mathematics to be used in the development of new theory. Most people find the need for the discovery of new mathematics hard to accept. Nonetheless, it is a fact that recent developments in mathematical logic, probability theory, statistics, game theory, modern algebra, topology, set theory and the mathematical theory of information have been essential to the understanding which does exist of electrical communication, language, computing devices, learning processes, servomechanisms, and many other of the communication sciences, and further mathematical developments are required if progress is to continue."

S. Sherman (Philadelphia, Pa.)

6955:

Fleischer, Isidore. The central concepts of communication theory for infinite alphabets. J. Math. Phys. 37 (1958), 223–228.

Let X be a probability space, **S** the class of its measur-

able sets, p its probability measure, and ϕ a subadditive (i.e., $\phi(A\cup B)\leqq\phi(A)+\phi(B)$) function defined on $\mathbf{S}$ and taking its value in the real numbers augmented with $+\infty$. With each finite partition ρ of $\mathbf{S}$ into measurable sets let $\sigma_{\mathbf{S}}{}^{\rho}(\phi)=\sum_{A\in\rho}\phi(A)$. Let $\int_{\mathbf{S}}\phi(dx)=\mathrm{df}\sup_{\phi}\sigma_{\mathbf{S}}{}^{\rho}(\phi)$. When $\phi(A)=-p(A)\log(A)$, then one gets $H(X, p)$ the uncertainty inherent in the probability space X. Theorem: If X is atomic, then $H(X, p)=-\sum_i p_i \log p_i$, where p_i are the probabilities of the atoms; if X is not atomic then $H(X, p)=+\infty$. Let X and Y be probability spaces with a joint distribution q on $X\times Y$, where X and Y represent the input and output alphabets on a channel whose action yields q. Let π be the (independent) product measure on $X\times Y$ of the two marginal distributions of q. When $\phi(A)=q(A)\log(q(A)/\pi(A))$ for measurable subsets A of $X\times Y$, then one gets the transmission rate

$$\int_{X\times Y} q(dx, dy)\log\frac{q(dx, dy)}{\pi(dx, dy)},$$

which is shown to be equal to $\int_{X\times Y}\log f(x, y)dq$, when q is absolutely continuous with respect to π and f its derivative, and to $+\infty$ otherwise. The theorem on transmission rate was stated without proof by Gelfand, Kolmogorov, and Yaglom [Dokl. Akad. Nauk SSSR **111** (1956), 745–748; MR **18**, 859].

S. Sherman (Philadelphia, Pa.)

6956:

Bourret, Richard. A note on an information theoretic form of the uncertainty principle. Information and Control 1 (1958), 398–401.

6957:

Zarechnak, Michael. Three levels of linguistic analysis in machine translation. J. Assoc. Comput. Mach. **6** (1959), 24–32.

This paper describes one of three independent methods used in Russian-to-English machine translation attempts by three different research units at Georgetown University.

"The goal of the general analysis method, under the author's direction, is to prove that a sentence can be handled by a machine in terms of its multi-layered constituents, so that the transfer of meaning can be adequately effected."

The author fails to state that this goal has not yet been attained, or to mention the previous existence of a commercially available translation (prepared without machines) of the Russian text chosen for his demonstration. The author's group plans to make a detailed analysis of a continuous corpus of 1000 sentences, in the hope that when the program is written to deal with the features encountered in this corpus, it can be extended later without radical changes to deal with other structural features.

The levels of linguistic analysis deal with individual words and endings (morphemic), relations between immediately adjacent words (syntagmatic), and solving such problems as locating the nuclei of verb phrases within sentences (syntactic). These are not claimed to be self-contained or independent.

E. F. Moore (Murray Hill, N.Y.)

CONTROL SYSTEMS

See also 6572.

6958:

Stiefel, E. Einführung in die Theorie der verallgemeinerten Funktionen (Distributionen) als mathematisches Werkzeug zur Behandlung linearer Regelungen. Bull. Schweiz. Elektrotechn. Vereins **1957**, no. 15, 3–8.

In this expository lecture the author outlines Mikusinski's theory of generalized functions of a non-negative variable, and explains the use of such generalized functions in the theory of linear servo-mechanisms.

A. Erdélyi (Pasadena, Calif.)

6959:

Badillo Barallat, M. C. New possibilities in trivalent logic and problems solved with relays and networks. Calc. Automat. y Cibernet. 7 (1958), no. 19, 28–39. (Spanish. French summary)

HISTORY AND BIOGRAPHY

6960:

Li, Shu-T'ien. Origin and development of the Chinese abacus. J. Assoc. Comput. Mach. 6 (1959), 102–110.

6961:

Tenca, Luigi. L'attività matematica di Evangelista Torricelli. Period. Mat. (4) 36 (1958), 251–263.

6962:

Errera, Alfred. Constantin Carathéodory. Rev. Univ. Bruxelles 2 (1958), 7 pp.

Selon l'auteur, 'Le présent article pourrait s'intituler "Carathéodory et la Belgique", car notre propos est surtout de mettre en relief ce que sa carrière scientifique dut à la formation qu'il avait reçue dans notre pays.'

6963:

Wang, Hao. The axiomatization of arithmetic. J. Symb. Logic 22 (1957), 145–158.

The author asks himself the question: "how were the famous axiom systems, such as Euclid's for geometry, Zermelo's for set theory, Peano's for arithmetic, originally obtained?" The main content of this article is the printing of a letter, dated 27 February, 1890, now in the Niedersächsische Staats- und Universitätsbibliothek at Göttingen, from Dedekind to a headmaster in Hamburg. In this letter, which shows the extent to which Peano borrowed, as he himself stated, from Dedekind, the notion of non-standard models (unintended interpretations) of axioms for positive integers is clearly brought out, with a discussion of the whole question of how, in the actual experience of a mathematician, an axiom system is gradually brought into its final form. The letter is of great historical interest.

The article closes with remarks about types of language in which the Dedekind (i.e. Peano) system is not categorical and some comments on the relation between Dedekind and Frege.

S. H. Gould (Providence, R.I.)

6964:

Babuška, Ivo; Havlíček, Karel; and Nožička, František. In memory of Prof. František Vyčichlo. Časopis Pěst. Mat. 83 (1958), 374–387. (1 plate) (Czech)

General and scientific biography, with bibliography.

6965:
Darwin, C. G. Obituary: Douglas Rayner Hartree. J. London Math. Soc. **34** (1959), 118–128.

A footnote states, "The Society gratefully acknowledges permission to reprint this notice from the Biographical Memoirs of the Royal Society (1958). A full bibliography will be found at the end of the original Memoir."

6966:
Todd, J. A. Obituary: John Hilton Grace. J. London Math. Soc. **34** (1959), 113–117.

A brief biography and a list of 39 publications.

6967:
Newman, M. H. A. Obituary: Hermann Weyl. J. London Math. Soc. **33** (1958), 500–511.

The author states, "This notice is a shortened form of one, written in collaboration with H. Davenport, P. Hall, G. E. H. Reuter and L. Rosenfeld, which appeared in the Biographical Memoirs of the Royal Society (1957). A full bibliography will be found at the end of that Memoir."

6968:
Rogosinski, W. W. Obituary: Michael Fekete. J. London Math. Soc. **33** (1958), 496–500.

A brief biography and a 77-item list of publications.

6969:
Gluskin, L. M.; and Lyapin, E. S. Anton Kazimirovič Suškevič (on his sixtieth birthday). Uspehi Mat. Nauk **14** (1959), no. 1(85), 255–260. (1 plate) (Russian)

A bibliographical sketch with one photograph and a bibliography with 71 entries.

6970:
★Eulerus, Leonhardus. Opera omnia. Series secunda. Opera mechanica et astronomica. Vol. VI. Commentationes mechanicae ad theoriam motus punctorum pertinentes. Vol. prius. Edidit Charles Blanc. Societas Scientiarum Naturalium Helveticae, Lausanne, 1957. xxxvii+302 pp.

The volume under review is the first of two collecting Euler's papers on the mechanics of mass-points. The preface by Blanc consists in short, clear summaries of the contents of the papers and enables the modern reader to see at once the domain to which the work belongs and to estimate the results obtained.

While these volumes are the least interesting among Euler's researches on mechanics, being concerned with the kind of "Analytical Mechanics" nowadays associated with examination problems of the last century and reducing, at bottom, to little more than investigation of explicitly integrable cases of certain ordinary differential equations, nevertheless a major paper on the principles of mechanics is included.

This is E86, published in 1746; it concerns the motion of bodies constrained to move on a rigid curve which itself may be in free or constrained motion. The posthumously pieces E826, E827, E828, and E829 contain earlier treatments of special cases. The problem was raised by John Bernoulli about 1730; a decade later, it attracted the attention of Daniel Bernoulli and Euler, and the latter communicated it to Clairaut. An early fruit, as Blanc remarks, was Euler's and Daniel Bernoulli's recognition of the angular momentum and proof that it is conserved in certain cases. More important is the realization by Euler and Clairaut of the nature and role of the principle of relative motion, which had major repercussions in discussions of invariance a century later.

An adequate history of the problem of relative motion has never been written. Clairaut [Mém. Acad. Sci. Paris **1742**, 1–52 (1745)] first achieved a correct general statement, in words, of the laws of mechanics in non-inertial frames, but his calculation of the relative acceleration in special cases is faulty. Daniel Bernoulli, characteristically, was able to solve very special problems correctly by special devices but made no attempt to face the general situation. Euler stood between. In the paper E86 he succeeded in solving, by a method which is general in principle but mathematically complex, some extremely difficult cases; for example, that of a particle mobile within a curved tube free in space. Application to the case when the tube is given an assigned motion foreshadows Euler's theory of rigid bodies.

For Euler's further development of the principle of relative motion, we must turn to his papers of 1751–1753 on hydraulics in Opera omnia, Ser. II, Vol. 15 [MR **19**, 826]. There he gives a complete and correct verbal statement, followed by a complete and correct mathematical theory of onedimensional motion expressed in the angular variables appropriate to rotating hydraulic machines. This brilliant analysis contains the first explicit appearance of the "Coriolis acceleration". However, Euler's attempt in 1755 to calculate this acceleration in three-dimensional rectangular Cartesian coordinates is faulty [Opera omnia, Ser. II, Vol. 12; MR **17**, 2; p. 109].

C. Truesdell (Bloomington, Ind.)

GENERAL

6971:
★Parke, Nathan Grier, III. Guide to the literature of mathematics and physics including related works on engineering science. 2nd revised ed. Dover Publications, Inc., New York, 1958. xviii+436 pp. $2.49.

The first edition [McGraw-Hill, New York-London, 1947] was reviewed in MR **8**, 496. "The number of entries has doubled, reaching more than 5000."

6972:
★James, Glenn; and James, Robert C. (Editors) Mathematics dictionary. 2nd ed., multilingual. D. Van Nostrand Co., Inc., Princeton-Toronto-New York-London, 1959. iv+546 pp. $15.00.

The first edition [Digest Press, Van Nuys, Calif., 1942] was reviewed in MR **3**, 293 and, in a revised form [1943], in MR **4**, 193. The present volume is an enlargement and revision of the second edition of 1949. Additions have been made, especially "in the fields of modern algebra, number theory, topology, vector spaces, the theory of games and linear and dynamic programming, numerical analysis, and computing machines". Appendices have been added, with tables, lists of symbols and French, German, Russian and Spanish indexes.

6973a:
★Труды третьего всесоюзного математического съезда, Москва, июнь-июль 1956. Том 1. Секционные доклады. [Proceedings of the third all-Union mathematical conference, Moscow, June-July, 1956. Vol. 1. Sectional reports.] Izdat. Akad. Nauk SSSR, Moscow, 1956. 238 pp. 14.70 rubles.

6973b:
★Труды третьего всесоюзного математического съезда, Москва, июнь-июль 1956. Том 2. Краткое содержание обзорных и секционных докладов. [Proceedings of the third all-Union mathematical conference, Moscow, June-July, 1956. Vol. 2. Summary of survey and sectional reports.] Izdat. Akad. Nauk SSSR, Moscow, 1956. 168 pp. 10 rubles.

6973c:
★Труды третьего всесоюзного математического съезда, Москва, июнь-июль 1956. Том 3. Обзорные доклады. [Proceedings of the third all-Union mathematical conference, Moscow, June-July, 1956. Vol. 3. Survey reports.] Izdat. Akad. Nauk SSSR, Moscow, 1956. 598 pp. 37.65 rubles.

6973d:
★Программа третьего всесоюзного математического съезда: Москва, 25 июня- 4 июля 1956г. [Program of the third all-Union mathematical conference: Moscow, 25 June-4 July, 1956.] Akad. Nauk SSSR, Ministerstvo Vysšego Obrazovaniya SSSR, Moscow, 1956. 98 pp.

The special sections of the conference were: theory of numbers, algebra, differential and integral equations, theory of functions, functional analysis, theory of probability, topology, geometry, mathematical logic and foundations of mathematics, numerical analysis, mathematical problems of mechanics, mathematical problems of physics, history of mathematics. The survey reports lasted 45 minutes, the sectional reports 30 or 20 minutes.

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